

Optimisation II - April 20

VIII) Inverse problems

Parameter identification problems:

We are measuring a state y_m , which we know to satisfy a parameter dependent PDE. Goal: find the parameters.

E.g.: y_m solves
$$\begin{cases} -\Delta y = u & \text{in } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

Given y_m , find u .

Straight forward solution: $u = -\Delta y_m$.

But:

- we have measurement errors:

$$y_m = \underbrace{y_t}_{\text{true state}} + \underbrace{n_s}_{\text{noise of noise level } \delta} (= y_s)$$

\leadsto would obtain $u = -\Delta y_s = \underbrace{-\Delta y_t}_{= u_t} - \Delta n_s$. true solution

- modelling errors

Classical finite dimensional approach: Try instead a

least squares solution:

$$\text{Solve } \min_{u, y} \frac{1}{2} \|y - y_\delta\|^2 \quad \text{s.t.} \quad \begin{cases} -\Delta y = a & \text{in } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

But: still no solution (in general) and: if a solution exists, it depends discontinuously on y_δ .

Solution: add "regularisation", e.g.:

$$\text{Solve } \min_{u, y} \frac{1}{2} \|y - y_\delta\|^2 + \frac{\alpha}{2} \|u\|^2 \quad \text{s.t.} \quad \begin{cases} -\Delta y = a & \text{in } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

\leadsto end up with standard linear-quadratic control problem.

$\alpha > 0$... "regularisation parameter"

For the following: more general setting:

U, V ... Hilbert spaces, $F: U \rightarrow V$ bounded linear

Given $y_\delta = y_t + n_\delta \in V$, find a solution of

$$Fu = y_\delta \quad (*)$$

We say that (*) is well-posed, if:

- it has a unique solution for all $y_\delta \in V$
- which depends continuously on y_δ .

Else, it is called ill-posed.

In the following: consider ill-posed problems
 \rightsquigarrow cannot solve (*) directly, but need
 to find an approximate solution that depends
 continuously on the data.

E.g.: (classical) Tikhonov regularization:

Solve

$$\min_u \left[\frac{1}{2} \|Fu - y_\delta\|^2 + \frac{\alpha}{2} \|u\|^2 \right] \quad (I)$$

for some regularization parameter $\alpha > 0$.

Theorem: The problem (T) has a unique solution $u_{\alpha, \delta}$ for all $y_\delta \in V$, and the solution depends continuously on y_δ .

Proof: Existence and uniqueness is standard.

Cont. dependence on y_δ : Assume that $y_k \rightarrow y_\delta$ and denote by u_k the solution of

$$\min_u \frac{1}{2} \|Fu - y_k\|^2 + \frac{\alpha}{2} \|u\|^2.$$

We have

$$\begin{aligned} \frac{1}{2} \|Fu_k - y_k\|^2 + \frac{\alpha}{2} \|u_k\|^2 &\leq \underbrace{\frac{1}{2} \|Fu_{\alpha, \delta} - y_k\|^2 + \frac{\alpha}{2} \|u_{\alpha, \delta}\|^2}_{\frac{1}{2} \|Fu_{\alpha, \delta} - y_\delta\|^2 + \frac{\alpha}{2} \|u_{\alpha, \delta}\|^2} \\ &\rightarrow \frac{1}{2} \|Fu_{\alpha, \delta} - y_\delta\|^2 + \frac{\alpha}{2} \|u_{\alpha, \delta}\|^2. \end{aligned}$$

This shows that the sequence (u_k) is bounded

and thus has a convergent subsequence $u_{k_i} \rightarrow \hat{u}$.

Since F is l.b. linear, we have that $Fu_{k_i} \rightarrow F\hat{u}$.

The mappings $u \mapsto \frac{\alpha}{2} \|u\|^2$, $u \mapsto \frac{1}{2} \|Fu - y_\delta\|^2$ are w.s.c.

Thus $\frac{1}{2} \|F\hat{u} - y_\delta\|^2 + \frac{\alpha}{2} \|\hat{u}\|^2 \leq \liminf_h \left[\frac{1}{2} \|Fu_k - y_\delta\|^2 + \frac{\alpha}{2} \|u_k\|^2 \right]$

$y_\delta = y_k - (y_k - y_\delta)$

$$= \liminf_h \left[\frac{1}{2} \|Fu_k - y_k\|^2 - \underbrace{\langle Fu_k - y_k, y_k - y_\delta \rangle}_{\text{bounded}} + \frac{1}{2} \|y_k - y_\delta\|^2 + \frac{\alpha}{2} \|u_k\|^2 \right] \rightarrow 0$$

$$= \liminf_h \left[\frac{1}{2} \|Fu_k - y_k\|^2 + \frac{\alpha}{2} \|u_k\|^2 \right]$$

$$\leq \liminf \left[\frac{1}{2} \|Fu - y_k\|^2 + \frac{\alpha}{2} \|u\|^2 \right] \quad \forall u \in \mathcal{U}$$

$$= \frac{1}{2} \|Fu - y_\delta\|^2 + \frac{\alpha}{2} \|u\|^2$$

Thus $\hat{u} = u_{\alpha, \delta}$.

[Still need that $u_k \rightarrow \hat{u}$, not only $u_k \rightarrow \hat{u}$]

have that

$$\frac{1}{2} \|Fu_{\alpha, \delta} - y_\delta\|^2 \leq \liminf_k \frac{1}{2} \|Fu_k - y_k\|^2$$

$$\frac{\alpha}{2} \|u_{\alpha, \delta}\|^2 \leq \liminf_k \frac{\alpha}{2} \|u_k\|^2$$

and [see **||**]

$$\frac{1}{2} \|Fu_{\alpha, \delta} - y_\delta\|^2 + \frac{\alpha}{2} \|u_{\alpha, \delta}\|^2 \geq \limsup \left[\frac{1}{2} \|Fu_k - y_k\|^2 + \frac{\alpha}{2} \|u_k\|^2 \right]$$

$$\left[\begin{array}{l} \text{Note: } a \leq \liminf_k a_k \\ b \leq \liminf_k b_k \\ (a+b) \geq \limsup_k (a_k + b_k) \end{array} \right] \Rightarrow \left. \begin{array}{l} a_k \rightarrow a \\ b_k \rightarrow b \end{array} \right]$$

$$\text{Thus: } \|u_k\|^2 \rightarrow \|u_{\alpha, \delta}\|^2$$

$$\begin{aligned} \text{Ans: } \|u_k - u_{\alpha, \delta}\|^2 &= \|u_k\|^2 - 2 \langle u_k, u_{\alpha, \delta} \rangle + \|u_{\alpha, \delta}\|^2 \\ &\rightarrow \|u_{\alpha, \delta}\|^2 - 2 \langle u_{\alpha, \delta}, u_{\alpha, \delta} \rangle + \|u_{\alpha, \delta}\|^2 \\ &\rightarrow 0 \end{aligned} \quad \square$$

Theorem: Assume that F is injective, that u_t solves

$$Fu = y_t, \text{ and let } y_k \in V \text{ be such that}$$

$$\delta_k = \|y_t - y_k\| \rightarrow 0 \text{ let moreover } \alpha_k \rightarrow 0$$

s.t. $\frac{\delta_k^2}{\alpha_k} \rightarrow 0$ and denote by u_k the solution

$$\text{of } \min_u \left[\frac{1}{2} \|Fu - y_k\|^2 + \frac{\alpha_k}{2} \|u\|^2 \right]$$

Then $u_k \rightarrow u_t$.

Proof: We have that

$$\begin{aligned}
\frac{\alpha_k}{2} \|u_n\|^2 &\leq \frac{1}{2} \|Fu_k - y_k\|^2 + \frac{\alpha_k}{2} \|u_k\|^2 \\
&\leq \frac{1}{2} \|Fu_t - y_k\|^2 + \frac{\alpha_k}{2} \|u_t\|^2 \\
&\quad \underbrace{=}_{=y_t} \\
&= \frac{\delta_k^2}{2} + \frac{\alpha_k}{2} \|u_t\|^2
\end{aligned}$$

Thus $\frac{1}{2} \|u_k\|^2 \leq \frac{\delta_k^2}{2\alpha_k} + \frac{1}{2} \|u_t\|^2$

$\underbrace{\hspace{10em}}_{\rightarrow 0}$

and (u_k) is bounded. \leadsto have a weakly conv. subsequence $u_{k'} \rightharpoonup \hat{u}$, and $Fu_{k'} \rightarrow F\hat{u}$.

Now: $\frac{1}{2} \|F\hat{u} - y_t\|^2 \leq \liminf_{k'} \frac{1}{2} \|Fu_{k'} - y_{k'}\|^2$

$$\begin{aligned}
&\leq \liminf_{k'} \left[\frac{1}{2} \|Fu_{k'} - y_{k'}\|^2 + \frac{\alpha_{k'}}{2} \|u_{k'}\|^2 \right] \\
&\leq \liminf_{k'} \left[\frac{1}{2} \underbrace{\delta_{k'}^2}_{\rightarrow 0} + \frac{\alpha_{k'}}{2} \underbrace{\|u_t\|^2}_{\rightarrow 0} \right] = 0
\end{aligned}$$

Thus $F\hat{u} = y_t \leadsto \hat{u} = u_t$.

Strong convergence $u_k \rightarrow u_t$ similar as before. \square

Parameter choice: DISCREPANCY

e.g.: discrepancy principle:

Choose α s.t. $\|F u_{\alpha, \delta} - y_{\delta}\| \approx \delta$