

Optimisation II - March 26

Continuation of proof:

Want to show existence for:

$$J(y, u) = \int_{\Omega} \varphi(x, y(x)) dx + \int_{\Gamma} \gamma(x, u(x)) ds \rightarrow \min$$

s.t. $u \in U_{ad}$ and (S) holds.

Have seen: (y_h, u_h) ... minimising sequence.

\rightsquigarrow the sequence (u_h) is bounded in L^2 , and
 (y_h) is bounded in H_0^1 .

\Rightarrow can choose weakly convergent subsequences

$$u_h \rightharpoonup \bar{u} \text{ in } L^2$$

$$y_h \rightharpoonup \bar{y} \text{ in } H_0^1$$

Since U_{ad} is closed and convex, it is weakly closed. Thus
 $\bar{u} \in U_{ad}$ as well.

Ans: have seen on Tuesday: the pair (\bar{y}, \bar{u}) solves
 the state equation. $\rightsquigarrow (\bar{y}, \bar{u})$ satisfies all the
 constraints.

Since ψ is convex in the second variable, the mapping
 $u \mapsto \int_{\Gamma_1}^{\Gamma_2} \psi(x, u(x)) ds$

is wsc in L^2 . Thus $\int_{\Gamma_1}^{\Gamma_2} \psi(x, \bar{u}(x)) ds \leq \liminf_{h \rightarrow \infty} \int_{\Gamma_1}^{\Gamma_2} \psi(x, y_h(x)) ds$.

Since $y_h \rightarrow \bar{y}$ in H^1 , we have $y_h \rightarrow y$ in L^2 .

Since ψ is non-negative, the mapping

$$y \mapsto \int_{\Omega} \psi(x, y(x)) dx$$

is lsc on L^2 . Thus $\int_{\Omega} \psi(x, \bar{y}(x)) dx \leq \liminf_{h \rightarrow \infty} \int_{\Omega} \psi(x, y_h(x)) dx$

Therefore $J(\bar{y}, \bar{u}) \leq \underbrace{\liminf_{h \rightarrow \infty} J(y_h, u_h)}_{= \inf_{y, u \text{ admissible}} J(y, u)}$

Thus (\bar{y}, \bar{u}) solves the control problem. \square

Example: Consider the problem ($\alpha > 0$)

$$\min_{y, u} \int_{\Omega} \sqrt{1 + (y(x) - y_2(x))^2} dx + \frac{\alpha}{2} \int_{\Gamma} u^2 ds$$

$$\text{s.t. } \begin{cases} y^3 - \Delta y = 0 & \text{in } \Omega \\ \partial_y y = u - y & \text{on } \Gamma \end{cases}$$

$\rightsquigarrow g, \psi$ are non-negative, ψ is convex, satisfies growth condition.

Have nonlinearity $d(x, y) = y^3$

\rightarrow require growth condition $|d(x, t)| \leq c_1 + c_2 |t|^{p-1}$

with $\begin{cases} 1 \leq p < \infty & \text{if } n=1, 2 \\ 1 \leq p < \frac{2n}{n-2} & \text{if } n=3, 4, \dots \end{cases}$

$$n=2 \quad \checkmark$$

$$n=3 \quad p < 6 \quad \checkmark$$

$$n=4 \quad p < 4 \quad \Rightarrow \text{does not work}$$

VI b) Fréchet differentiability and optimality conditions

Consider again the reduced functional

$$f(u) = J(S(u), u)$$

$$f : L^2(\Gamma_1) \rightarrow \mathbb{R}$$

with $S : L^2(\Gamma_1) \rightarrow L^2(\Omega)$ the solution

operator for the PDE.

If J and S are Fréchet differentiable, then

$$\nabla f(u) = S(u)^* \begin{bmatrix} \nabla_y J(S(u), u) \\ \nabla_u J(S(u), u) \end{bmatrix}$$

Moreover:

$$\nabla_y J(S(u), u) = \partial_y g(x, S(u)(x))$$

$$\nabla_u J(S(u), u) = \partial_u \gamma(x, u(x))$$

Ans: (see Ch. ∇f) J is Fréchet-differentiable if y, γ are continuously differentiable in the second variable, and

$$|\partial_y g(x, t)| \leq c_1 + c_2 |t|$$

$$|\partial_u \gamma(x, t)| \leq c_1 + c_2 |t|$$

Problem: Fréchet differentiability of S

Theorem: (Inverse function theorem)

Let Y, Z be Banach spaces and $F: Y \rightarrow Z$ be continuously Fréchet differentiable. Let $y_0 \in Y$ and $F'(y_0) \in L(Y, Z)$ is boundedly invertible, that is, $F'(y_0)$ is bijective and $F'(y_0)^{-1}$ is bounded linear. Then there exist neighborhoods U of y_0 ,

and) V of $F(y_0)$ s.t. $\bar{F}: U \rightarrow V$ is bijection

Moreover $\bar{F}^{-1}: V \rightarrow U$ is continuously Fréchet
Differentiable with $(\bar{F}^{-1})'(z) = F'(\bar{F}(z))^{-1}$.

Now apply this to S or rather:

Regard the PDE as solution of an equation

$A(y) = \text{right hand side}$

with $A: H_{T_0}^1 \rightarrow (H_{T_0}^1)^*$

$$A(y) := \left[v \mapsto \int \limits_{\Omega} d(x, y(x)) v(x) dx + a(y, v) \right]$$

Want to show that it satisfies the conditions of
inverse function theorem.

• Fréchet differentiability:

the mapping $y \mapsto a(y, \cdot)$ is bounded linear.

\rightarrow only need to show that $G: H_{T_0}^1 \rightarrow (H_{T_0}^1)^*$

$$G(y) \mapsto \left[v \mapsto \int \limits_{\Omega} d(x, y(x)) v(x) dx \right]$$

is Fréchet differentiable \Rightarrow Decompose G as

$$G = \gamma \circ \underline{\Phi} \circ i$$

with $i : H_{\gamma}^1 \rightarrow L^p$ with $\begin{cases} 1 \leq p < \infty & \text{if } n=1,2 \\ 1 \leq p \leq \frac{2n}{n-2} & \text{if } n \geq 3 \end{cases}$

$$y \mapsto y$$

$$\bullet \quad \underline{\Phi} : L^p \rightarrow L^q \quad \text{with } q = \frac{p}{p-1}$$

the superposition operator $y \mapsto d(x, y(x))$

$$\bullet \quad j : L^q \rightarrow (H_{\gamma}^1)^*$$

\Downarrow

$$z \mapsto \left[v \mapsto \int_S z v \, dx \right]$$

Then i, j are bounded linear and thus Fréchet differentiable.

(Chap. II F) $\Rightarrow \underline{\Phi}$ is Fréchet differentiable with

$$\underline{\Phi}'(y)z = \partial_y d(x, y(x)) z(x)$$

if d is continuously differentiable in x and

$$(*) \quad |\partial_y d(x, t)| \leq C_1 + C_2 |t|^{(\frac{p}{q}-1)} = p-2$$

Thus: If $(*)$ holds G and therefore f is Fréchet

$\zeta =: f(z, v)$

$$A'(y)z = \left[v \mapsto \boxed{\int_{\Omega} \underbrace{\partial_y d(x, y(x)) z(x)}_{\geq 0} v(x) dx + a(z, v)} \right]$$

- * $A'(y)$ is (?) boundedly invertible:

That is: The equation $A'(y)z = \ell$ has a solution for every $\ell \in (H^1_{\Gamma_0})^*$, which depends continuously on ℓ . Or: The PDE

$f(z, v) = \ell(v) \quad \forall v \in H^1_{\Gamma_0}$
 has a unique solution for all $\ell \in (H^1_{\Gamma_0})^*$ which depends continuously on ℓ .

But: f_ℓ is a bounded bilinear form. Ans:

$$\begin{aligned} f(z, z) &= \int_{\Omega} \underbrace{\partial_y d(x, y(x))}_{\geq 0 \text{ since } d \text{ is increasing in } y} z(x)^2 dx + a(z, z) \\ &\geq g(z, z) \\ &\geq c \|z\|_{H^1_{\Gamma_0}}^2 \end{aligned}$$

since a is coercive $\Rightarrow f_\ell$ is bounded
 and coercive \Rightarrow Lax-Milgram implies unique

separability + continuous dependence of RHS.

\Rightarrow conditions of inverse function theorem holds.

\Rightarrow have a Fréchet differentiable solution operator.