Norwegian University of Science
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Department of Mathematical
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TMA4183
Optimisation II
Spring 2020

Exercise set 4

1 For the following variational problems, determine whether they admit a solution and, if yes, whether the solution is unique. Moreover, derive the associated Euler–Lagrange equation and discuss its solvability. (Note that this may depend on the dimension of the set Ω.)

Here

$$X := \{u \in H^1(\Omega) : u|_\Gamma = g\},$$

where we always assume that X is non-empty, that is, that there exists a function $u \in H^1(\Omega)$ with $u|_\Gamma = g$.

a) The problem

$$\int_\Omega \frac{1}{4}u^4 + \frac{1}{2}|\nabla u(x)|^2\,dx \to \min_{u \in X}.$$

b) The problem

$$\int_\Omega c(x)|\nabla u(x)|^2\,dx \to \min_{u \in X}.$$

Here $c \in C^1(\bar{\Omega})$ satisfies $c(x) > 0$ for all $x \in \bar{\Omega}$.

c) The problem

$$\frac{1}{2}\int_\Omega (1 + u(x)^2)|\nabla u(x)|^2\,dx \to \min_{u \in X}.$$

2 Assume that $f \in C^1(\bar{\Omega})$ is a fixed, given function such that $f(x) < 0$ for all $x \in \partial\Omega$. Define moreover

$$X := \{u \in H_0^1(\Omega) : u(x) \geq f(x) \text{ for a.e. } x \in \Omega\}.$$

Show that the *obstacle problem*

$$\min_{u \in X} \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx$$

admits a unique solution.

Try to find optimality conditions for this problem!