Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4183 Optimisation II Spring 2020

Exercise set 1

1 The *Cantor set* is constructed by the following approach: We start with the interval $[0, 1]$ and remove from it the central open interval $(1/3, 2/3)$. This results in the set $[0, 1/3] \cup [2/3, 1]$, which is the union of two disjoint closed intervals. From each of those intervals, we then remove again their central parts, that is, the intervals $(1/9, 2/9)$ and $(7/9, 8/9)$, and end up with the union of four disjoint intervals of length $1/9$. Again, we remove the central part of each of the subintervals and obtain a union of eight disjoint intervals of length $1/27$. This process of always removing the central part of each subinterval is then continued *ad infinitum* and the resulting set is called the *Cantor set*, denoted in the following by C.

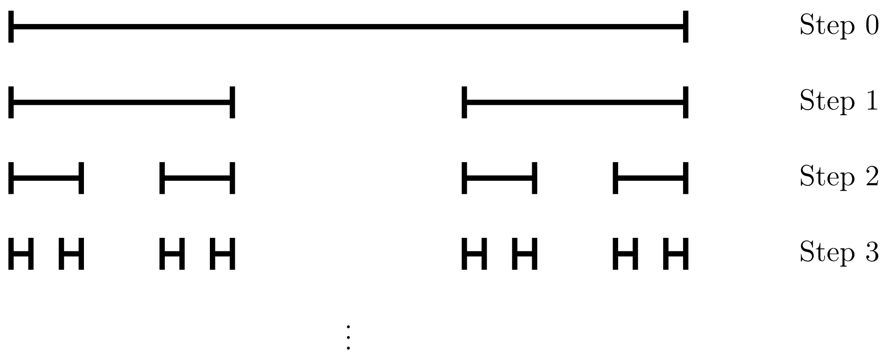

Figure 1: Sketch of the construction of the Cantor set.

a) Show that C is a closed set and that $\mathcal{L}^1(C) = 0$.

b) Show that the set C consists precisely of the reals in the unit interval that have an expansion in base 3, where none of the digits is equal to 1. In other words, show that

$$C = \Big\{ x \in [0, 1] : x = \sum_{k=1}^{\infty} a_k 3^{-k} \text{ with } a_k \in \{0, 2\} \text{ for all } k \Big\}.$$

c) Show that the mapping $f : C \to [0, 1]$ defined by

$$f\Big(\sum_{k=1}^{\infty} a_k 3^{-k}\Big) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k}$$

is surjective, and conclude that the cardinality of C is the same as the cardinality $\mathfrak{c}$ of the reals.

In particular, this shows that there exists uncountable sets of measure zero.

- *Possible solution:*

  a) Denote by C_k the set that is obtained at the k-th step of the construction. That is, $C_0 = [0,1]$, $C_1 = [0,1/3] \cup [2/3,1]$, ... Then the set C_k is the disjoint union of 2^k closed intervals of length 3^{-k}, and $C = \bigcap_{k\in\mathbb{N}} C_k$.
  Since C_k is the finite union of closed sets, it is closed. As a consequence, the set C is the intersection of (infinitely many) closed sets, and therefore closed as well.
  Moreover, $\mathcal{L}^1(C_k) = \frac{2^k}{3^k}$, as C_k is just a disjoint union of 2^k intervals of length 3^{-k}. Since $C \subset C_k$ for all k, it follows that
  $$\mathcal{L}^1(C) \le \mathcal{L}^1(C_k) = \frac{2^k}{3^k}$$
  for all k, and therefore $\mathcal{L}^1(C) = 0$.

  b) Define $D_1 = (1/3, 2/3)$, $D_2 = (1/9,2/9) \cup (3/9,4/9) \cup (5/9,6/9) \cup (7/9,8/9)$, and more general
  $$D_k = \bigcup_{k=1}^{(3^k-1)/2} \Big(\frac{2k-1}{3^k}, \frac{2k}{3^k}\Big).$$
  Then we can write
  $$C_\ell = [0,1] \setminus \Big(\bigcup_{k=1}^{\ell} D_k\Big)$$
  and
  $$C = [0,1] \setminus \Big(\bigcup_{k\in\mathbb{N}} D_k\Big).$$
  Next we note that the set D_1 consists of the reals for which the first digit in ternary expansion is equal to 1, with the exception of the points 1/3 and 2/3. These points, however, have the ternary expansions
  $$1/3 = 0.0222\ldots, \qquad \text{and} \qquad 2/3 = 0.2000\ldots.$$
  Thus the complementary set $[0,1] \setminus D_1$ consists of precisely those reals that can be written in ternary expansion with either a 0 or a 2 at the first digit. Similarly, the set $[0,1] \setminus D_k$ consists of those reals that can be written in ternary expansion with either a 0 or a 2 at the k-th digit. Since C is the intersection of all these sets, the claimed representation of C follows.

  c) The surjectivity of the mapping f follows immediately from the fact that every real has a binary expansion. Since the cardinality of $[0,1]$ is equal to $\mathfrak{c}$ and $f\colon C \to [0,1]$ is surjective, it follows that the cardinality of C is at least $\mathfrak{c}$. On the other hand, as $C \subset [0,1]$ it is at most $\mathfrak{c}$ as well, which proves the claim.[1]

2 *(Reverse Hölder inequality)*

Assume that $0 < p < 1$ and denote by q the Hölder conjugate exponent of p, that is, $q = p/(p-1)$ (note that $q < 0$!). Let moreover u, $v\colon E \to \mathbb{R}$ be measurable functions such that $u(x) \ge 0$ and $v(x) > 0$ for almost every $x \in E$. Show that
$$\int_E uv\,dx \ge \Big(\int_E u^p\,dx\Big)^{1/p}\Big(\int_E v^q\,dx\Big)^{1/q}.$$

[1] There is a bit more to that: Since every subset of a negligible set is Lebesgue measurable (and C is negligible), it follows that every subset of C is Lebesgue measurable. As a consequence, the cardinality of the class of Lebesgue measurable sets is equal to $2^{\mathfrak{c}}$. However, one can show that the cardinality of the class of sets that can be written as countable intersection and/or union of intervals (such sets are called *Borel sets*) is only $\mathfrak{c}$. As a consequence, there exist (many...) Lebesgue measurable sets that cannot be written as countable union and/or union of intervals.

Hint: Write $u^p = (uv)^p v^{-p}$ and apply the Hölder inequality to $\int_E (uv)^p v^{-p}\, dx$.

- *Possible solution:* As indicated in the hint, we write $u^p = (uv)^p v^{-p}$. Moreover, we set $r := 1/p$ and denote by s the Hölder conjugate exponent to r, that is,

$$s = \frac{s}{s-1} = \frac{1}{p} \cdot \frac{1}{\frac{1}{p} - 1} = -\frac{q}{p}.$$

Then the standard Hölder inequality (with exponents s and r) implies that (since both u and v are non-negative, we can ignore all absolute values)

$$\int_E u^p\, dx = \int (uv)^p v^{-p}\, dx \leq \Big(\int_E \big((uv)^p\big)^r\, dx \Big)^{1/r} \Big(\int_E (v^{-p})^s\, dx \Big)^{1/s}$$
$$= \Big(\int_E uv\, dx \Big)^p \Big(\int_E v^q\, dx \Big)^{-\frac{p}{q}}.$$

Taking p-th roots on both sides and multiplying by $(\int_E v^q\, dx)^{1/q}$, we arrive at the desired inequality.

3 Assume that $1 \leq p < q \leq +\infty$. Show that $L^p([0,1]) \not\subseteq L^q([0,1])$.

- *Possible solution:* Assume $1 \leq p < q < \infty$ (the case $q = +\infty$ will be treated later). We need find a function u such that

$$\int_0^1 |u(x)|^p\, dx < \infty \qquad \text{and} \qquad \int_0^1 |u(x)|^q\, dx = \infty.$$

The maybe simplest choice here is the function

$$u(x) = \frac{1}{x^{1/q}}.$$

We have

$$\int_0^1 |u(x)|^q\, dx = \int_0^1 \frac{1}{x}\, dx = +\infty,$$

whereas

$$\int_0^1 |u(x)|^p\, dx = \int_0^1 x^{-p/q}\, dx = \frac{1}{1-p/q}\, x^{1-p/q}\Big|_0^1 = \frac{1}{1-p/q}.$$

For $p \geq 1$ and $q = +\infty$, we can for instance choose

$$u(x) = \frac{1}{x^{2/p}},$$

which is obviously unbounded and thus not contained in $L^\infty([0,1])$, whereas

$$\int_0^1 |u(x)|^p\, dx = \int_0^1 x^{-1/2}\, dx = \frac{1}{2}\, x^{1/2}\Big|_0^1 = \frac{1}{2}.$$

4 Assume that $1 \leq p < q \leq +\infty$. Show that $L^q(\mathbb{R}) \not\subseteq L^p(\mathbb{R})$.

- *Possible solution:* For $q = +\infty$ we can choose the function $u \equiv 1$, which is obviously bounded and thus contained in $L^\infty(\mathbb{R})$, but $\int u(x)^p \, dx = \infty$ for all $1 \leq p < \infty$.

  For $1 \leq p < q < +\infty$, we may choose

$$u(x) = \begin{cases} 1/x^{1/p} & \text{if } x \geq 1, \\ 0 & \text{else.} \end{cases}$$

  Then

$$\int_{\mathbb{R}} |u(x)|^p \, dx = \int_1^\infty \frac{1}{x} \, dx = +\infty,$$

  whereas

$$\int_{\mathbb{R}} |u(x)|^q \, dx = \int_1^\infty x^{-q/p} \, dx = \frac{1}{1-q/p} \, x^{1-q/p} \Big|_1^\infty = -\frac{1}{1-q/p}.$$