[Tr] = Fredi Tröltzsch, "Optimal control of PDEs: Theory, Methods, and Applications"

1 Lecture 1

Chapter 1: Introduction, examples of optimal control problems, basic concepts in finite-dimensional case.

Keywords: control, state, solution operator (control-to-state operator), reduced cost functional, reduced problem, reduced gradient, adjoint state, adjoint problem, first order necessary optimality conditions, formal Lagrange method, minimizing sequence

2 Lecture 2

Section 2.1: Normed and inner product spaces and their complete versions (Banach and Hilbert spaces); examples of function spaces (C^k, L^p)

Section 2.4: Linear functionals.

Keywords: dual space, bidual, reflexive spaces.

Main results: projection onto a closed convex set (below), Riesz representation theorem as its consequence (Exercise set 1).

Theorem 1. *Let H be a Hilbert space, $C \subseteq H$ be a non-empty closed subset, and $\hat{x} \in H$. Then there is a unique point $\bar{x} \in C$ such that*

$$\|\hat{x} - \bar{x}\|^2 = \inf_{y \in C} \|\hat{x} - y\|^2. \tag{1}$$

Proof. Strategy: (1) show existence; (2) show uniqueness. (1.1) Take a minimizing sequence for (1); (1.2) Show that it is Cauchy; (1.3) Its limit is a solution.

To simplify the notation we assume $\hat{x} = 0$ (this corresponds to translating the coordinate system in H to $\hat{x}$ and does not change any properties of C or (1)).

(1.1) Let $0 \leq I := \inf_{y \in C} \|y\|^2$ and select a minimizing sequence $\{y_k\}_{k=1}^{\infty}$, $y_i \in C$, that is $I = \lim_{k \to \infty} \|y_k\|^2$.

(1.2) Using the fact that the norm in H is given by the inner product we can write, $\forall a, b \in H$:

$$\|a \pm b\|^2 = (a \pm b, a \pm b) = \|a\|^2 + \|b\|^2 \pm 2(a, b),$$

and therefore

$$\|a - b\|^2 = 2\|a\|^2 + 2\|b\|^2 - \|a + b\|^2, \quad \text{or}$$
$$\frac{1}{2}\|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2\left\|\frac{a + b}{2}\right\|^2.$$

Now we apply this equality to our minimizing sequence:

$$0 \le \frac{1}{2}\|y_n - y_m\|^2 = \|y_n\|^2 + \|y_m\|^2 - 2\left\|\frac{y_n + y_m}{2}\right\|^2 \le \|y_n\|^2 + \|y_m\|^2 - 2I,$$

where the last inequality holds because $(y_n + y_m)/2 \in C$ (C is a convex set) and $I = \inf_{y\in C}\|y\|^2$. Thus the right hand side of this inequality converges to 0 when $n, m \to \infty$ and therefore the sequence $\{y_k\}$ is Cauchy.

(1.3) H is a complete space therefore there is $\bar{x} \in H$ such that $\bar{x} = \lim_{k\to\infty} y_k$. Since C is closed we have that $\bar{x} \in C$. Since $\|\cdot\|$ is a continuous function, it holds that $I = \lim_{k\to\infty}\|y_k\|^2 = \|\bar{x}\|^2$ and as result $\|\bar{x}\|^2 = \min_{y\in C}\|y\|^2$.

(2) Uniqueness: let $\bar{x}_1, \bar{x}_2 \in C$ be such that $I = \|\bar{x}_1\|^2 = \|\bar{x}_2\|^2$ yet $\bar{x}_1 \neq \bar{x}_2$. Then $\bar{x}_1 + \bar{x}_2 \in C$ and therefore

$$I \le \left\|\frac{\bar{x}_1 + \bar{x}_2}{2}\right\|^2 = \frac{1}{2}\left(\|\bar{x}_1\|^2 + \|\bar{x}_2\|^2 - \frac{1}{2}\|\bar{x}_1 - \bar{x}_2\|^2\right) < I,$$

which is a contradiction. □

3 Lecture 3

Section 2.4.2: Weak convergence in Banach spaces.

Main result: Theorem 2.10 (every bounded sequence in a reflexive Banach space contains a weakly converging subsequence).

We have proved a slightly weaker result:

Theorem 2. *Let X be a reflexive Banach space, and assume that its dual X' is separable; namely let $\{f_i\}_{i=1}^{\infty}$ be a countable everywhere dense subset in X'. Let $\{x_k\}_{k=1}^{\infty}$ be a bounded sequece in X. Then there is a subsequence $\{x'_k\}$ of $\{x_k\}$, which converges weakly in X.*

Proof. Diagonal process.

Let $M > 0$ be an upper bound on the norms of x_k, i.e. $\forall k : \|x_k\| \le M$.

Then the sequence of real numbers $\{f_1(x_k)\}_{k=1}^{\infty}$ is bounded; namely $|f_1(x_k) \le \|f_1\|_{X'}\|x_k\|_X \le M\|f_1\|_{X'}$. Therefore it contains a convergent subsequence (this is Heine-Borel theorem). Let us denote the corresponding subsequence of $\{x_k\}$ as $\{x_k^{(1)}\}$.

Similarly the sequence of real numbers $\{f_2(x_k^{(1)})\}_{k=1}^{\infty}$ is bounded; namely $|f_2(x_k^{(1)}) \le \|f_2\|_{X'}\|x_k\|_X \le M\|f_2\|_{X'}$. Therefore it contains a convergent subsequence. Let us denote the corresponding subsequence of $\{x_k^{(1)}\}$ as $\{x_k^{(2)}\}$.

We can proceed like this indefinitely. Let us consider the *diagonal subsequence*: $\{x_k^{(k)}\}_{k=1}^{\infty}$.

Note that the "tail" of the diagonal sequence, $\{x_k^{(k)}\}_{k=N}^{\infty}$ is a subsequence of $\{x_k^{(i)}\}_{k=1}^{\infty}$ for all $i \leq N$. As a consequence of this, the sequence of real numbers $\{f_i(x_k^{(k)})\}_{k=1}^{\infty}$ is Cauchy for all $i = 1, 2, \ldots$. Let us define a function $F(f_i) = \lim_{k\to\infty} f_i(x_k^{(k)})$. Clearly F is linear, and it is also bounded: $\forall i$, $|F(f_i)| \leq M\|f_i\|_{X'}$.

First will show that F can be extended to be an element of X'' (we can do this directly without using Hanh-Banach theorem in the present situation). Indeed, take an arbitrary $f \in X'$, and consider an arbitrary sequence $\{f_i\}_{i=1}^{\infty}$ converging to f where f_i are elements of our everywhere dense subset. Then

$$|F(f_i) - F(f_j)| = |F(f_i - f_j)| \leq \lim_{k\to\infty} |(f_i - f_j)(x_k^{(k)})| \leq M\|f_i - f_j\|_{X'}.$$

Thus the sequence of numbers $\{F(f_i)\}$ is Cauchy and we can define $F(f) = \lim_{i\to\infty} F(f_i)$. Furthermore, $|F(f)| \leq M\|f\|_{X'}$ for all $f \in X'$ and therefore $F \in X''$.

Since X is reflexive, there is $\bar{x} \in X$ such that $\forall f \in X' : F(f) = f(\bar{x})$. We will now show that $x_k^{(k)} \rightharpoonup \bar{x}$. Let $\varepsilon > 0$ be arbitrary. For an arbitrary $f \in X'$ let us select and fix f_i from the everywhere dense set such that $\|f - f_i\|_{X'} \leq \varepsilon$. We now use a classical 3ε-type argument:

$$\begin{aligned}|f(\bar{x} - x_k^{(k)})| &\leq |(f - f_i)(\bar{x})| + |(f - f_i)(x_k^{(k)})| + |f_i(\bar{x}) - f_i(x_k^{(k)})| \\ &\leq \varepsilon\|\bar{x}\|_X + \varepsilon M + |F(f_i) - f_i(x_k^{(k)})|.\end{aligned}$$

The last term becomes arbitrarily small when we let $k \to \infty$. Thus $\forall f \in X'$ we have $f(x) = \lim_{k\to\infty} f(x_k^{(k)})$. □

4 Lecture 4

Section 2.4.2: Weak convergence, closedness, and compactness in Banach spaces.

Keywords: epigraph of a function, lower-semicontinuity

Main results: Hanh–Banach separation theorem. Theorems 2.11 and 2.12. Weierstrass' theorem (existence of optimal solutions):

Theorem 3. *Let X be a reflexive Banach space, $C \subset X$ be nonempty, closed, bounded, and convex; and $f : X \to \mathbb{R}$ be lower semi-continuous and convex. Then there is $\bar{x} \in C$ such that*

$$f(\bar{x}) = \inf_{x\in C} f(x).$$

Proof. Sketch: (1) Take a minimizing sequence $x_k \in C$; (2) C is sequentially weakly compact (Theorem 2.11) therefore (up to a subsequence) $x_k \rightharpoonup \hat{x} \in$

C; (3) f is weakly sequentially lower semi-continuous (Theorem 2.12) , and therefore

$$\inf_{x \in C} f(x) \leq f(\bar{x}) \leq \lim_{k \to \infty} f(x_k) = \inf_{x \in C} f(x).$$

□

5 Lecture 5

Section 2.2: Sobolev spaces.

Section 2.3: Weak solutions to elliptic equations.

Keywords: Weak derivative, trace theorem (Theorem 2.1), regular (Lipschitz) domain, Lax-Milgram lemma (Lemma 2.2), Friedrichs inequality (Lemma 2.3)

Note that we have considered a version of Lax–Milgram lemma, where we have assumed that the bilinear form is *symmetric*. In this case the bilinear form defines an equivalent inner product on our space, and existence of solutions to the variational statement is given by Riesz representation theorem.