TMA4183 Opt. II
Spring 2017

**Exercise set 7**

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Please read sections 4.3–4.4 in [Tr].

1 Exercise 4.4 (ii) in [Tr]: Show that Nemytskii operator $y(\cdot) \mapsto \sin(y(\cdot))$ is Frechet differentiable from $L^{p_1}(0,T)$ into $L^{p_2}(0,T)$ whenever $1 \le p_2 < p_1 \le \infty$.

Hint: convergence in $L^{p_1}(0,T)$ implies convergence in measure; that is if $\|h_n\|_{L^{p_1}(0,T)} \to 0$ then for any $\varepsilon > 0$: $\mathcal{L}(\{\, x \in (0,T) : |h_n(x)| > \varepsilon \,\}) \to 0$ where $\mathcal{L}$ is the Lebesgue measure (think of an "area") of the set.

**Solution:** Let us put $\Psi(y) = \sin(y(\cdot))$. The directional derivative

$$\Psi'(y;h) = \lim_{\varepsilon \downarrow 0} \frac{\Psi(y+\varepsilon h) - \Psi(y)}{\varepsilon} = \cos(y(\cdot))h(\cdot),$$

is linear with respect $h$.

Thus we need to check that $\|\Psi(y+h) - \Psi(y) - \Psi'(y;h)\|_{L^{p_2}(0,T)}/\|h\|_{L^{p_1}(0,T)} \to 0$ when $\|h\|_{L^{p_1}(0,T)} \to 0$.

Let $\{h_n\}_{n=1}^{\infty} \in L^{p_1}(0,T)$ be a sequence converging to zero, and let us put $r_n = \Psi(y+h_n) - \Psi(y) - \Psi'(y;h_n)$. Additionally, let us chose an arbitrary small $\varepsilon > 0$. We will divide $\Omega = (0,T)$ into two parts: $\omega_n = \{\, x \in \Omega \mid |h_n(x)| < \varepsilon \,\}$ and $\Omega_n = \Omega \setminus \omega_n$. We will denote the characteristic functions of $\omega_n$ (respectively, $\Omega_n$) with $\chi_{\omega_n}$ (resp. $\chi_{\Omega_n}$.

On $\omega_n$ we can use the second order Taylor series expansion of $\sin(\cdot)$ combined with the fact that the second derivative of $\sin(\cdot)$ is bounded by 1 to get the estimate $|r_n(x)| \le |h_n(x)|^2/2 \le \varepsilon|h_n(x)|/2$.

On the other hand, $\Omega_n$ will be very small in measure for large $n$ (since convergence to zero in $L^{p_1}(0,T)$ implies convergence in measure). Furthermore, on $\Omega_n$ we can write $|r_n(x)| \le |\sin(y(x)+h_n(x)) - \sin(y(x))| + |\cos(y(x))h_n(x)| \le 2|h_n(x)|$, because $\sin(\cdot)$ is a Lipschitz function with constant 1 (derivative is bounded by 1).

Thus on $\omega_n$ we use the Taylor's series and Hölder's inequality to get the estimate:

$$\begin{aligned}
\|\chi_{\omega_n} r_n\|_{L^{p_2}(0,T)} &\le \varepsilon/2 \|\chi_{\omega_n} h_n\|_{L^{p_2}(0,T)} = \varepsilon/2 \bigg( \int_0^T \chi_{\omega_n}(x)|h_n(x)|^{p_2}\,\mathrm{d}x \bigg)^{1/p_2} \\
&\le \varepsilon/2 \big( \|\chi_{\omega_n}\|_{L^{p_1/(p_1-p_2)}(0,T)} \||h_n|^{p_2}\|_{L^{p_1/p_2}(0,T)} \big)^{1/p_2} \\
&= \varepsilon/2 |\omega_n|^{(p_1-p_2)/(p_1 p_2)} \|h_n\|_{L^{p_1}(0,T)},
\end{aligned}$$

where $|\omega_n|$ denotes the Lebesgue measure of $\omega_n$, which is bounded by $|\Omega| = T$ in our case.

Similarly, on $\Omega_n$ we get

$$\|\chi_{\Omega_n} r_n\|_{L^{p_2}(0,T)} \le 2\|\chi_{\Omega_n} h_n\|_{L^{p_2}(0,T)} \quad \le 2|\Omega_n|^{(p_1-p_2)/(p_1p_2)}\|h_n\|_{L^{p_1}(0,T)}.$$

In summary,

$$\begin{aligned}\|r_n\|_{L^{p_2}(0,T)} &= \|\chi_{\omega_n} r_n + \chi_{\Omega_n} r_n\|_{L^{p_2}(0,T)} \le \|\chi_{\omega_n} r_n\|_{L^{p_2}(0,T)} + \|\chi_{\Omega_n} r_n\|_{L^{p_2}(0,T)} \\ &\le (\varepsilon|\Omega|^{(p_1-p_2)/(p_1p_2)}/2 + 2|\Omega_n|^{(p_1-p_2)/(p_1p_2)})\|h_n\|_{L^{p_1}(0,T)},\end{aligned}$$

and therefore

$$0 \le \lim_{n\to\infty} \frac{\|r_n\|_{L^{p_2}(0,T)}}{\|h_n\|_{L^{p_1}(0,T)}} \le \varepsilon|\Omega|^{(p_1-p_2)/(p_1p_2)}/2,$$

since $\lim_{n\to\infty} |\Omega_n|^{(p_1-p_2)/(p_1p_2)} = 0$ for an arbitrary $\varepsilon > 0$, ***because*** $p_1 > p_2$. It remains to let $\varepsilon \to 0$ in the last inequality.

2 Compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ (Rellich–Kondrachov Theorem, Theorem 7.4 in [Tr]) plays an important role in the proof of Theorem 4.15 (existence of optimal controls for semi-linear elliptic PDEs). There are many other examples of compact embeddings.

Let $-\infty < a < b < +\infty$, and consider the spaces of continuous functions $C^0[a,b]$ and Hölder continuous functions $C^{0,\gamma}[a,b]$, $0 < \gamma \le 1$. These spaces are equipped with the norms

$$\begin{aligned}\|f\|_{C^0[a,b]} &= \sup_{x\in[a,b]} |f(x)|, \\ \|f\|_{C^{0,\gamma}[a,b]} &= \|f\|_{C^0[a,b]} + \sup_{x\neq y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.\end{aligned}$$

We will use Arzela–Ascoli characterization of relative compactness in $C^0[a,b]$ (it is not difficult to prove either) The set $S \subset C^0[a,b]$ is relatively compact (i.e. a set whose closure is compact) if and only if it is *bounded* and *equicontinuous*. That is, there is $M > 0$ such that $\forall f \in S : \|f\|_{C^0[a,b]} \le M$, and for every $\varepsilon > 0$ there is $\delta > 0$: $\forall f \in S, x, y \in [a,b] : |x-y| < \delta \implies |f(x)-f(y)| < \varepsilon$.

**a)** Show that $C^{0,\gamma}[a,b]$ is continuously embedded into $C^0[a,b]$.

**Solution:** Per definition of $\|\cdot\|_{C^{0,\gamma}[a,b]}$: $\forall f \in C^{0,\gamma}[a,b]$

$$\|f\|_{C^0[a,b]} \le \|f\|_{C^{0,\gamma}[a,b]} = \|f\|_{C^0[a,b]} + \text{something non-negative}.$$

Therefore the operator $i : C^{0,\gamma}[a,b] \to C^0[a,b]$ defined by $i(f) = f$ is linear and bounded.

**b)** Show that every bounded subset in $C^{0,\gamma}[a,b]$ is bounded and equicontinuous in $C^0[a,b]$. Conclude that from any bounded sequence in $C^{0,\gamma}[a,b]$ one can extract a subsequence, which is Cauchy in $C^0[a,b]$.

**Solution:** Assume that $S \subset C^{0,\gamma}[a,b]$ is such that $\exists M > 0 : \forall f \in S, \|f\|_{C^{0,\gamma}[a,b]} \le M$. By definition $\|f\|_{C^0[a,b]} \le \|f\|_{C^{0,\gamma}[a,b]} \le M$ and thus $S$ is also a bounded set in $C^0[a,b]$. Furthermore from the definition of the norm we have that $|f(x)-f(y)| \le |x-y|^\gamma \|f\|_{C^{0,\gamma}[a,b]}$. Thus as long as $|x-y| < \delta$ it follows that $\forall f \in S : |f(x)-f(y)| < \delta^\gamma M$. Thus is is sufficient to choose $\delta = (\varepsilon/M)^{1/\gamma}$ in the definition of equicontinuity.

**c)** Let $V_1$, $V_2$ be two Banach spaces, and assume that $V_1$ is continuously embedded into $V_2$. Show that $V_2'$ is continuously embedded into $V_1'$ if we simply consider restrictions of functionals in $V_2$ onto $V_1$.
Conclude that if $v_k \rightharpoonup \bar{v}$, weakly in $V_1$ then also $v_k \rightharpoonup \bar{v}$, weakly in $V_2$.

**Solution:** Let $f \in V_2'$, and define $g : V_1 \to \mathbb{R}$ as a restriction of $f$ onto $V_1$ That is, for all $v \in V_1$ we have $g(v) = f(v) = f(i(v)) = i'(f)(v)$, $v \in V_1$, where $i : V_1 \to V_2$ is the continuous embedding map. Then clearly $g = i'(f)$ is linear and bounded, since $i' : V_2' \to V_1'$ is the adjoint of a bounded linear operator. Alternatively one can estimate the norm of $g$ directly:

$$|g(v)| = |f(v)| \leq \|f\|_{V_2'}\|v\|_{V_2} \leq \|f\|_{V_2'}\|v\|_{V_1},$$

where the last inequality is owing to the continuous embedding of $V_1$ into $V_2$. Therefore $\|g\|_{V_1'} \leq \|f\|_{V_2'}$, and in this sense $V_2'$ is continuously embedded into $V_1'$.
Assume now that $v_k \rightharpoonup \bar{v}$ in $V_1$. Then, for all $g \in V_1'$: $\lim_{k\to\infty} g(v_k - \bar{v}) = 0$. By the previous discussion the restrictions of $f \in V_2'$ onto $V_1$ are in $V_1'$ and therefore for all $f \in V_2'$: $\lim_{k\to\infty} f(v_k - \bar{v}) = 0$.

**d)** Show that any sequence $f_n \in C^{0,\gamma}[a,b]$, which converges weakly to some limit $\bar{f} \in C^{0,\gamma}[a,b]$, must satisfy $\|f_n - \bar{f}\|_{C^0[a,b]} \to 0$.
Hint: weakly convergent sequences are bounded (uniform boundedness principle); weak limit is unique (consequence of Hanh–Banach theorem); use the proof by contradiction and **a)–c)**.

**Solution:** Suppose that $f_n \rightharpoonup \bar{f} \in C^{0,\gamma}[a,b]$. Weakly convergent sequences are bounded (uniform boundedness principle), and thus the set $S := \{f_n, n = 1, 2, \dots\}$ is relatively compact in $C^0[a,b]$ according to **a)–b)** and Arzela-Ascolli theorem.
Finally, assume that $\|f_n - \bar{f}\|_{C^0[a,b]} \not\to 0$, that is, for some $\varepsilon > 0$ there is a subsequence $n'$ of $n$ such that $\|f_{n'} - \bar{f}\|_{C^0[a,b]} \geq \varepsilon$. Since $\{f_{n'}\}$ is a sequence in $S$, a relatively compact set in $C^0[a,b]$, we can extract a further subsequence $n''$ from it, such that $\|f_{n''} - \tilde{f}\|_{C^0[a,b]} \to 0$, for some $\tilde{f} \in C^0[a,b]$.
Thus we end up with a sequence $f_{n''}$ with the following properties. First, $f_{n''}$ converges weakly to $\bar{f}$ in $C^0[a,b]$ (because it is a subsequence of $f_n$; $f_n \rightharpoonup \bar{f}$ in $C^{0,\gamma}[a,b]$ and finally because of **a), c)**). Second, $f_{n''}$ converges weakly to $\tilde{f}$ in $C^0[a,b]$ (in fact is even converges strongly to $\tilde{f}$).
Owing to the assumptions on $n'$, we have $\tilde{f} \neq \bar{f}$. Thus the subsequence $f_{n''}$ has two distinct weak limits in $C^0[a,b]$. This contradicts the uniqueness of the weak limit (consequence of Hahn–Banach theorem).