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TMA4183 Opt. II
Spring 2017

Exercise set 7

Please read sections 4.3–4.4 in [Tr].

1 Exercise 4.4 (ii) in [Tr]: Show that Nemytskii operator $y(\cdot) \mapsto \sin(y(\cdot))$ is Frechet differentiable from $L^{p_1}(0,T)$ into $L^{p_2}(0,T)$ whenever $1 \leq p_2 < p_1 \leq \infty$.

Hint: convergence in $L^{p_1}(0,T)$ implies convergence in measure; that is if $\|h_n\|_{L^{p_1}(0,T)} \to 0$ then for any $\varepsilon > 0$: $\mathcal{L}(\{ x \in (0,T) : |h_n(x)| > \varepsilon \}) \to 0$ where $\mathcal{L}$ is the Lebesgue measure (think of an "area") of the set.

2 Compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ (Rellich–Kondrachov Theorem, Theorem 7.4 in [Tr]) plays an important role in the proof of Theorem 4.15 (existence of optimal controls for semi-linear elliptic PDEs). There are many other examples of compact embeddings.

Let $-\infty < a < b < +\infty$, and consider the Banach spaces of continuous functions $C^0[a,b]$ and Hölder continuous functions $C^{0,\gamma}[a,b]$, $0 < \gamma \leq 1$. These spaces are equipped with the norms

$$\|f\|_{C^0[a,b]} = \sup_{x\in[a,b]} |f(x)|,$$
$$\|f\|_{C^{0,\gamma}[a,b]} = \|f\|_{C^0[a,b]} + \sup_{x\neq y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.$$

We will use Arzela–Ascoli characterization of relative compactness in $C^0[a,b]$ (it is not difficult to prove either) The set $S \subset C^0[a,b]$ is relatively compact (i.e. a set whose closure is compact) if and only if it is *bounded* and *equicontinuous*. That is, there is $M > 0$ such that $\forall f \in S : \|f\|_{C^0[a,b]} \leq M$, and for every $\varepsilon > 0$ there is $\delta > 0$: $\forall f \in S, x, y \in [a,b] : |x-y| < \delta \implies |f(x)-f(y)| < \varepsilon$.

a) Show that $C^{0,\gamma}[a,b]$ is continuously embedded into $C^0[a,b]$.

b) Show that every bounded subset in $C^{0,\gamma}[a,b]$ is bounded and equicontinuous in $C^0[a,b]$. Conclude that from any bounded sequence in $C^{0,\gamma}[a,b]$ one can extract a subsequence, which is Cauchy in $C^0[a,b]$.

c) Let V_1, V_2 be two Banach spaces, and assume that V_1 is continuously embedded into V_2. Show that V_2' is continuously embedded into V_1' if we simply consider restrictions of functionals in V_2 onto V_1.
Conclude that if $v_k \rightharpoonup \bar{v}$, weakly in V_1 then also $v_k \rightharpoonup \bar{v}$, weakly in V_2.

d) Show that any sequence $f_n \in C^{0,\gamma}[a,b]$, which converges weakly to some limit $\bar{f} \in C^{0,\gamma}[a,b]$, must satisfy $\|f_n - \bar{f}\|_{C^0[a,b]} \to 0$.
Hint: weakly convergent sequences are bounded (uniform boundedness principle); weak limit is unique (consequence of Hanh–Banach theorem); then use the proof by contradiction and **a)–c)**.