Exercise set 6

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Please read sections 4.1–4.2 in [Tr]. In this exercise we will prove the existence part in a slightly weaker version of Browder–Minty's theorem. The proof utilizes the following topological result (Brower's fixed point theorem).

Theorem 1. *Let C be a compact convex set and $f : C \to C$ a continuous function. Then there is at least one point $x \in C$ such that $f(x) = x$ (fixed point of f).*

1 Consider the following standard "counter-example" to Brower's fixed point theorem in Hilbert spaces. Let H be a Hilbert space with an orthonormal basis $\{ e_1, e_2, \dots \}$. Let B be the closed unit ball in H and and consider a map $f : B \to B$ sending a vector x with coordinates $(x_1, x_2, \dots)$ into a vector y with coordinates $((1 - \|x\|_H^2)^{1/2}, x_1, x_2, \dots)$.

a) Show that $f : B \to B$ is continous.

Solution: From here on we will write vectors in terms of their coordinates.
We can represent $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = ((1 - \|x\|_H^2)^{1/2}, 0, 0, \dots)$ and $f_2(x) = (0, x_1, x_2, \dots)$.
f_1 is a composition of the continuous functions (norm, square root) and a linear map $i : \mathbb{R} \to H$ defined as $i(r) = (r, 0, 0, \dots)$. The latter map is an isometry (and therefore is bounded/continuous).
f_2 is also a linear isometry (and therefore is bounded/continuous).
It is easy to check that $\|f(x)\|_H^2 = (1 - \|x\|_H) + \sum_i |x_i|^2 = 1$, and therefore $f(B) \subset B$.

b) Show that f has no fixed points in B.

Solution: If $f(x) = x$ then $x_2 = x_1$, $x_3 = x_2$, $\dots$, and as a result $x_i = x_1$, $i \geq 1$. The only possible vector with "constant" coordinates in H is 0 (since $\|x\|_H^2 = \sum_i |x_i|^2 < \infty$). However $f(0) = (1, 0, \dots) \neq 0$.

c) Which of the assumptions of Browder's fixed point theorem is violated by this example?

Solution: B is closed, convex, but *not compact* in an infinite-dimensional Hilbert space.

2 Let V be a reflexive separable Banach space, $\{w_1, w_2, \dots\}$ be everywhere dense in

V. Let further $A : V \to V'$ be a monotone, coercive, hemicontinuous, and *bounded*[1] operator. Then for every $f \in V'$ there is $y \in V$ such that $Ay = f$.

We start with some auxiliary results about A.

a) Use hemicontinuity of A to show that if $0 \le (Av - f)(v - y)$ for all $v \in V$ then $Ay = f$.

Solution: Let $v = y + tw$, where $w \in V$ is arbitrary. Then

$$0 \le (A(y + tw) - f)(y + tw - y) = t[(A(y + tw) - f)(w)].$$

Since $\pm w \in V$ the inequality must actually be an equality. Finally, we use hemicontinuity to observe that $(A(y + tw))(w) \to (A(y))(w)$ as $t \to 0$.

b) Assume that $\lim_{n\to\infty} \|y_n - y\|_V = 0$. Show that $\forall v \in V$: $\lim_{n\to\infty}(A(y_n))(v) = (A(y))(v)$, that is, $A(y_n) \rightharpoonup A(y)$ in V'.[2] Hint: use monotonicity and the previous characterization of $Ay = f$.

Solution: Since the sequence $\{y_n\}$ is bounded in V and the operator A is bounded, the sequence $\{A(y_n)\}$ is bounded in V'. Since V' is a reflexive Banach space, the latter sequence contains a weakly converging subsequence, let us say $A(y_{n'}) \rightharpoonup g \in V'$. We want to show that $g = A(y)$, or equivalently that $\forall v \in V$: $0 \le (Av - g)(v - y)$. Indeed, let $v \in V$ be arbitrary. Then, owing to the monotonicity:

$$0 \le (Av - Ay_{n'})(v - y_{n'}) = (Av)(v) - (Ay_{n'})(v) - (Av)(y_{n'}) + (Ay_{n'})(y_{n'}).$$

The last term on the right hand side converges to $g(y)$:

$$\begin{aligned}|(Ay_{n'})(y_{n'}) - g(y)| &\le |(Ay_{n'})(y_{n'} - y)| + |(Ay_{n'} - g)(y)| \\ &\le \|Ay_{n'}\|_{V'}\|y_{n'} - y\|_V + |(Ay_{n'} - g)(y)| \to 0,\end{aligned}$$

where we used the boundedness of $\{Ay_{n'}\}$, convergence $\|y_{n'} - y\|_V \to 0$, and finally the weak convergence $Ay_{n'} \rightharpoonup g$. Returning to the previous inequality this implies that

$$0 \le (Av - Ay_{n'})(v - y_{n'}) \to (Av)(v) - g(v) - (Av)(y) + g(y) = (Av - g)(v - y).$$

It remains to show that the convergence $A(y_n) \rightharpoonup A(y)$ happens along the whole sequence, not just a subsequence n'. This follows from the following standard argument: asume that there is no convergence along some sequence (i.e., $\exists \varepsilon > 0$, $v \in V$, n'': $\liminf_{n''\to\infty} |(A(y_{n''}) - A(y))(v)| \ge \varepsilon$). Then this subsequence n'' satisfies all the assumptions that the original sequence does. Therefore from it we can extract a further subsequence, say n''' such that $A(y_{n'''}) \rightharpoonup A(y)$, which is a contradiction.

The remainder of the existence proof utilizes Galerkin method: let $V_n = \operatorname{span}(w_1, w_2, \dots, w_n) \subset V$, $f_n \in V_n'$ is the restriction of f to V_n, and similarly $A_n : V_n \to V_n'$ be defined as $V_n \ni v \mapsto A(v)$ restricted to V_n. We can describe this process in more details by selecting a basis in V_n.

Namely, let $\{e_1, e_2, \dots, e_m\} \in V_n$, $m \le n$ be a basis in V_n. Define $i_n : \mathbb{R}^m \to V$ as $i_n((x_1, \dots, x_m)) = \sum_{k=1}^m x_k e_k$. Let further $i_n' : V' \to \mathbb{R}^m$ be defined as $(i_n'(f))(x) = f(i_n(x))$, or in other words $\sum_{k=1}^m [i_n'(f)]_k x_k = \sum_{k=1}^m x_k f(e_k)$.

[1] $A : V \to V'$ is bounded if for every bounded set $S \subset V$ there is a constant $C_S > 0$ such that $\forall v \in S : \|Av\|_{V'} \le C_S$.

[2] Such operators A are called *demicontinuous*.

c) Show that the problem of finding $y_n \in V_n$ such that $A_n y_n = f_n$ in V_n' is equivalent to finding $x \in \mathbb{R}^m$ such that $i_n'(A(i_n(x)) - f) = 0$ in $\mathbb{R}^m$.

Solution: Since $\{e_1, \dots, e_m\}$ is a basis in V_n, the problem of finding $y_n \in V_n$ such that $A_n y_n = f_n$ is equivalent to finding $x \in \mathbb{R}^m$ such that $A_n(i_n(x)) = f_n$. Note that $V_n' \ni g = 0$ iff $g(v) = 0$, $\forall v \in V_n$ iff $g(i_n(x)) = 0$, $\forall x \in \mathbb{R}^m$ iff $i_n'(g) = 0$ in $\mathbb{R}^m$. This brings us to the new equivalent problem $i_n'(A_n(i_n(x)) - f_n) = 0$. Finally, note that $i_n'(f_n) = f_n(i_n(x)) = f(i_n(x)) = i_n'(f)$ and a similarly $i_n'(A_n(i_n(x)) = i_n'(A(i_n(x))$, per definition of f_n and A_n.

d) Let us define a function $F_n : \mathbb{R}^m \to \mathbb{R}^m$ by $F_n(x) = i_n'(A(i_n(x)) - f)$. Show that F_n is continuous.

Solution: Suppose that $\mathbb{R}^m \ni x_k \to \bar{x} \in \mathbb{R}^n$. Since $i_n : \mathbb{R}^m \to V$ is linear and bounded, we have $v_k = i_n(x_k) \to i_n(\bar{x}) = \bar{v}$. Owing to the *demicontinuity* of A, we have $Av_k \rightharpoonup A\bar{v}$ in V'. Finally, i_n' has finite rank and as such maps weakly converging sequences into strongly converging ones. Indeed, for all $1 \le j \le m$:

$$[i_n'(Av_k)]_j = (Av_k)(e_j) \to (A\bar{v})(e_j) = [i_n'(A\bar{v})]_j.$$

e) Use coercivity of A and continuity of F_n to show that for some $r > 0$ if $\|x\|_{\mathbb{R}^m} \ge r$ then the product $x^{\mathrm{T}} F_n(x) \ge 0$.

Solution: Consider the product $x^{\mathrm{T}} F_n(x) = (A(i_n(x) - f)(i_n(x))$. Let x_k be the point of minimum attained by the continuous function $x^{\mathrm{T}} F_n(x)$ on the sphere $S_k := \{\, x \in \mathbb{R}^m \mid \|x\|_{\mathbb{R}^m} = k \,\}$ (the minimum is attained since the sphere is a compact set in $\mathbb{R}^m$). Then $\|x_k\|_{\mathbb{R}^m} = k \to \infty$ as $k \to \infty$.
We now note that $\|i_n(x)\| \ge \|i_n^{-1}\|_{\mathcal{L}(V_n,\mathbb{R}^m)} \|x\|_{\mathbb{R}^m}$ and therefore $\|i_n(x)\|_V \to \infty$ when $\|x\|_{\mathbb{R}^m} \to \infty$. Since A is coercive, $(A(i_n(x))(i_n(x))/\|i_n(x)\|_V \to \infty$ when $\|x\|_{\mathbb{R}^m} \to \infty$, whereas $|f(i_n(x))|/\|i_n(x)\|_V \le \|f\|_{V'}$.
As a consequence, we must have $\inf_{x\in S_k} x^{\mathrm{T}} F_n(x) \ge x_k^{\mathrm{T}} F_n(x_k) \ge 0$ for all $k \ge K$.

f) Use Brower's fixed point theorem to show that for every $n = 1, 2, \dots$ the problem $F_n(x) = 0$ admits a solution $x_n \in \mathbb{R}^m$ (hence also $A_n y_n = f_n$ where $y_n = i_n(x_n)$) by considering fixed points of a map $x \mapsto -rF_n(x)/\|F_n(x)\|_{\mathbb{R}^m}$ of the ball $B_r := \{\, x \in \mathbb{R}^m \mid \|x\|_{\mathbb{R}^m} \le r \,\}$ into itself (in fact, into its boundary), where $r > 0$ is found in the previous part.

Solution: Assume that $F_n(x) \ne 0$, $\forall x \in \mathbb{R}^m$. Then the function $g_n(x) := -rF_n(x)/\|F_n(x)\|_{\mathbb{R}^m}$ is continuous and maps the convex compact set B_r into itself (in fact, into S_r). Therefore, there must be a fixed point $\hat{x} \in B_r$ such that $\hat{x} = -rF_n(\hat{x})/\|F_n(\hat{x})\|_{\mathbb{R}^m} \in S_r$. Consider now the product $\hat{x}^{\mathrm{T}} F_n(\hat{x}) = -r\|F_n(\hat{x})\|_{\mathbb{R}^m} < 0$. According to the previous part it must be nonnegative, which brings us into a contradiction with the assumption that $F_n(x) \ne 0$, $\forall x \in \mathbb{R}^m$.

g) Use coercivity of A to show that the sequence $\{y_n\}$ is bounded in V.

Solution: Suppose that $\|y_n\| \to \infty$. Then

$$\begin{aligned} +\infty = \lim_{n\to\infty} \frac{(Ay_n)(y_n)}{\|y_n\|_V} &= \lim_{n\to\infty} \frac{(A_n y_n)(y_n)}{\|y_n\|_V} = \lim_{n\to\infty} \frac{f_n(y_n)}{\|y_n\|_V} \\ &= \lim_{n\to\infty} \frac{f(y_n)}{\|y_n\|_V} \le \lim_{n\to\infty} \frac{\|f\|_{V'}\|y_n\|_V}{\|y_n\|_V}, \end{aligned}$$

which is a contradiction.

Since $\{y_n\}$ is a bounded sequence and V is a reflexive Banach set, it contains a weakly converging subsequence, say $y_{n'} \rightharpoonup \bar{y}$. Similarly, since A is bounded then the sequence $\{Ay_{n'}\}$ is bounded in V' (which is a reflexive Banach set in its own right) and therefore also contains a weakly converging subsequence $Ay_{n''} \rightharpoonup g \in V'$ (and still $y_{n''} \rightharpoonup \bar{y}$).

h) Use the separability of V to show that $g = f$.

Solution: Let us take an arbitrary $v \in V$. Owing to separability of V we can find a sequence $v_n \in V_n$ such that $\lim_{n\to\infty} \|v_n - v\|_V = 0$. Then we can write:

$$\begin{aligned}
g(v) &= \lim_{n''\to\infty} (Ay_{n''})(v) = \lim_{n''\to\infty} (Ay_{n''})(v_{n''}) + (Ay_{n''})(v - v_{n''}) \\
&= \lim_{n''\to\infty} (A_{n''}y_{n''})(v_{n''}) + (Ay_{n''})(v - v_{n''}) = \lim_{n''\to\infty} f_{n''}(v_{n''}) + r_{n''} \\
&= \lim_{n''\to\infty} f(v_{n''}) + r_{n''} = f(v),
\end{aligned}$$

where the last equality holds owing to the continuity of f (recall $f \in V'$) if we can show that $r_{n''} \to 0$.
To estimate $r_{n''}$ we write:

$$\lim_{n''\to\infty} |r_{n''}| \leq \lim_{n''\to\infty} \|Ay_{n''}\|_{V'} \|v - v_{n''}\|_V = 0,$$

owing to the fact that the first factor is bounded (the sequence $y_{n''}$ is bounded and the operator A is bounded) and the last factor converges to 0.

i) Utilize the previously established convergence(s) and monotonicity of A to show that for an arbitrary $v \in V$ we have the inequality $0 \leq (Av - f)(v - \bar{y})$ (and as a result, $A(\bar{y}) = f$).

Solution: Owing to the monotonicity of A:

$$\begin{aligned}
0 &\leq (Av - Ay_{n''})(v - y_{n''}) = (Av)(v) - (Ay_{n''})(v) - (Av)(y_{n''}) + A(y_{n''})(y_{n''}) \\
&= (Av)(v) - (Ay_{n''})(v) - (Av)(y_{n''}) + A_n(y_{n''})(y_{n''}) \\
&= (Av)(v) - (Ay_{n''})(v) - (Av)(y_{n''}) + f_n(y_{n''}) \\
&= (Av)(v) - (Ay_{n''})(v) - (Av)(y_{n''}) + f(y_{n''}) \\
&\to (Av)(v) - f(v) - (Av)(\bar{y}) + f(\bar{y}) = (Av - f)(v - \bar{y}).
\end{aligned}$$