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# TMA4183 Opt. II
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**Exercise set 6**

Please read sections 4.1–4.2 in [Tr]. In this exercise we will prove the existence part in a slightly weaker version of Browder–Minty's theorem. The proof utilizes the following topological result (Brower's fixed point theorem).

**Theorem 1.** *Let $C$ be a compact convex set and $f : C \to C$ a continuous function. Then there is at least one point $x \in C$ such that $f(x) = x$ (fixed point of $f$).*

**1** Consider the following standard "counter-example" to Brower's fixed point theorem in Hilbert spaces. Let $H$ be a Hilbert space with an orthonormal basis $\{ e_1, e_2, \dots \}$. Let $B$ be the closed unit ball in $H$ and and consider a map $f : B \to B$ sending a vector $x$ with coordinates $(x_1, x_2, \dots)$ into a vector $y$ with coordinates $((1 - \|x\|_H^2)^{1/2}, x_1, x_2, \dots)$.

**a)** Show that $f : B \to B$ is continous.

**b)** Show that $f$ has no fixed points in $B$.

**c)** Which of the assumptions of Browder's fixed point theorem is violated by this example?

**2** Let $V$ be a reflexive separable Banach space, $\{w_1, w_2, \dots\}$ be everywhere dense in $V$. Let further $A : V \to V'$ be a monotone, coercive, hemicontinuous, and *bounded*[1] operator. Then for every $f \in V'$ there is $y \in V$ such that $Ay = f$.

We start with some auxiliary results about $A$.

**a)** Use hemicontinuity of $A$ to show that if $0 \leq (Av - f)(v - y)$ for all $v \in V$ then $Ay = f$.

**b)** Assume that $\lim_{n\to\infty} \|y_n - y\|_V = 0$. Show that $\forall v \in V$: $\lim_{n\to\infty}(A(y_n))(v) = (A(y))(v)$, that is, $A(y_n) \rightharpoonup A(y)$ in $V'$.[2] Hint: use monotonicity and the previous characterization of $Ay = f$.

The remainder of the existence proof utilizes Galerkin method: let $V_n = \operatorname{span}(w_1, w_2, \dots, w_n) \subset V$, $f_n \in V_n'$ is the restriction of $f$ to $V_n$, and similarly $A_n : V_n \to V_n'$ be defined as $V_n \ni v \mapsto A(v)$ restricted to $V_n$. We can describe this process in more details by selecting a basis in $V_n$.

[1] $A : V \to V'$ is bounded if for every bounded set $S \subset V$ there is a constant $C_S > 0$ such that $\forall v \in S : \|Av\|_{V'} \leq C_S$.

[2] Such operators $A$ are called *demicontinuous*.

Namely, let $\{e_1, e_2, \dots, e_m\} \in V_n$, $m \leq n$ be a basis in $V_n$. Define $i_n : \mathbb{R}^m \to V$ as $i_n((x_1, \dots, x_m)) = \sum_{k=1}^m x_k e_k$. Let further $i_n' : V' \to \mathbb{R}^m$ be defined as $(i_n'(f))(x) = f(i_n(x))$, or in other words $\sum_{k=1}^m [i_n'(f)]_k x_k = \sum_{k=1}^m x_k f(e_k)$.

**c)** Show that the problem of finding $y_n \in V_n$ such that $A_n y_n = f_n$ in $V_n'$ is equivalent to finding $x \in \mathbb{R}^m$ such that $i_n'(A(i_n(x)) - f) = 0$ in $\mathbb{R}^m$.

**d)** Let us define a function $F_n : \mathbb{R}^m \to \mathbb{R}^m$ by $F_n(x) = i_n'(A(i_n(x)) - f)$. Show that $F_n$ is continuous.

**e)** Use coercivity of $A$ and continuity of $F_n$ to show that for some $r > 0$ if $\|x\|_{\mathbb{R}^m} \geq r$ then the product $x^{\mathrm{T}} F_n(x) \geq 0$.

**f)** Use Brower's fixed point theorem to show that for every $n = 1, 2, \dots$ the problem $F_n(x) = 0$ admits a solution $x_n \in \mathbb{R}^m$ (hence also $A_n y_n = f_n$ where $y_n = i_n(x_n)$) by considering fixed points of a map $x \mapsto -rF_n(x)/\|F_n(x)\|_{\mathbb{R}^m}$ of the ball $B_r := \{\, x \in \mathbb{R}^m \mid \|x\|_{\mathbb{R}^m} \leq r \,\}$ into itself (in fact, into its boundary), where $r > 0$ is found in the previous part.

**g)** Use coercivity of $A$ to show that the sequence $\{y_n\}$ is bounded in $V$.

Since $\{y_n\}$ is a bounded sequence and $V$ is a reflexive Banach set, it contains a weakly converging subsequence, say $y_{n'} \rightharpoonup \bar{y}$. Similarly, since $A$ is bounded then the sequence $\{Ay_{n'}\}$ is bounded in $V'$ (which is a reflexive Banach set in its own right) and therefore also contains a weakly converging subsequence $Ay_{n''} \rightharpoonup g \in V'$ (and still $y_{n''} \rightharpoonup \bar{y}$).

**h)** Use the separability of $V$ to show that $g = f$.

**i)** Utilize the previously established convergence(s) and monotonicity of $A$ to show that for an arbitrary $v \in V$ we have the inequality $0 \leq (Av - f)(v - \bar{y})$ (and as a result, $A(\bar{y}) = f$).