Exercise set 5

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Please read section 2.15 in [Tr]. Note that the regularity of optimal controls relies upon the preservation of the weak differentiability by the projection map $\mathbb{P}_{[u_a,u_b]}(u) = \max\{u_a, \min\{u_b, u\}\}$. Since $\max\{u_1, u_2\} = (|u_1 - u_2| + u_1 + u_2)/2$, this issue hinges upon the preservation of the weak differentiability by the absolute value map.

1 Let Ω be an open set in $\mathbb{R}^N$. We will prove the following fact: if $u \in H^1(\Omega)$ then $|u| \in H^1(\Omega)$.

For $\epsilon > 0$ we will use the following regularization (approximation) of $|\cdot|$: $f_\epsilon(t) = (t^2 + \epsilon^2)^{1/2}$.

a) Let $\epsilon > 0$. Show that f_ϵ is a Lipschitz continuous function with Lipschitz constant 1.

Proof:

$$|f_\epsilon(t_1) - f_\epsilon(t_2)| = |f_\epsilon'(\tau)(t_1 - t_2)|,$$

where τ is between t_1 and t_2. Finally $|f_\epsilon'(\tau)| = |\tau/(\tau^2 + \epsilon^2)^{1/2}| \le 1$.

The second "trick" is the density of "nice" functions, for example $C^1(\Omega)$, in $H^1(\Omega)$. For any $u_k \in H^1(\Omega) \cap C^1(\Omega)$ and any $\phi \in C_0^\infty(\Omega)$ we have the equality

$$\int_\Omega f_\epsilon(u_k) D_i \phi = -\int_\Omega \phi f_\epsilon'(u_k) D_i u_k.$$

Furthermore, for any $u \in H^1(\Omega)$ there is a sequence of $u_k \in H^1(\Omega) \cap C^1(\Omega)$ such that that $\lim_{k\to\infty} \|u_k - u\|_{H^1(\Omega)} = 0$. Per definition, it means that both the function values and its derivatives converge in $L^2(\Omega)$, and thus converge almost everywhere pointwise for some subsequence. We relabel $\{u_k\}$ to be this subsequence. We now want to show that both sides of the integral equality above are continuous with respect to this type of convergence.

b) Use the Lipschitz continuity of f_ϵ to show that

$$\lim_{k\to\infty} \int_\Omega |(f_\epsilon(u_k) - f_\epsilon(u)) D_i \phi| = 0.$$

Proof: Indeed, claim follows from the Lipschitz continuity of f_ϵ + C-S inequality.

c) Show that

$$\lim_{k\to\infty}\int_\Omega |\phi[f_\epsilon'(u_k)D_i u_k - f_\epsilon'(u)D_i u| = 0.$$

Proof:

$$\begin{aligned}\int_\Omega &|\phi[f_\epsilon'(u_k)D_i u_k - f_\epsilon'(u)D_i u]| \\ &\le \int_\Omega |\phi[f_\epsilon'(u_k)D_i u_k - f_\epsilon'(u_k)D_i u]| + \int_\Omega |\phi[f_\epsilon'(u_k)D_i u - f_\epsilon'(u)D_i u]| \\ &\le \int_\Omega |\phi[D_i u_k - D_i u]| + \int_\Omega |\phi[f_\epsilon'(u_k) - f_\epsilon'(u)]D_i u|,\end{aligned}$$

where we used the fact that $|f_\epsilon'(u_k(x))| \le 1$. The first integral converges to zero because $\|D_i u_k - D_i u\|_{L^2(\Omega)} \to 0$. In the second integral we use the dominated Lebesgue convergence theorem. Indeed, $f_\epsilon'(u_k) \to f_\epsilon'(u)$, pointwise. Furthemore, the integrand is bounded by an integrable function $2|\phi D_i u|$, where again the inequality $|f_\epsilon'(\cdot)| \le 1$ is employed.

At this point we know that $\forall u \in H^1(\Omega)$, $\phi \in C_0^\infty(\Omega)$ we have the equality

$$\int_\Omega f_\epsilon(u) D_i\phi = -\int_\Omega \phi f_\epsilon'(u) D_i u.$$

We now let $\epsilon \to 0$, show that both sides of the equality converge, and identify the limits.

d) Show that

$$\lim_{\epsilon\to\infty}\int_\Omega |[f_\epsilon(u) - |u|]D_i\phi| = 0.$$

Proof: Dominated Lebesgue convergence theorem is applicable. Indeed, we have pointwise convergence. The bound can be established as follows. First we note that the function $t \mapsto |t|^{1/2}$ is Lipschitz continuous with constant $1/2$ on the sets $t \ge 1$ and $t \le -1$ (bound on the first derivative). Therefore if $|u| \ge 1$ then $0 < f_\epsilon(u) - |u| = f_\epsilon(u) - (u^2)^{1/2} \le \epsilon^2/2$. If $|u| \le 1$ then $0 < f_\epsilon(u) - |u| < (\epsilon^2+1)^{1/2}$. In either case, the integrand is bounded by the integrable function $\max\{(\epsilon^2+1)^{1/2}, \epsilon^2/2\}|D_i\phi|$.

e) Finally, show that

$$\lim_{\epsilon\to 0}\int_\Omega \phi|f_\epsilon'(u) - \operatorname{sign} u|D_i u = 0.$$

Proof: Dominated Lebesgue convergence theorem applies with the bound $2|\phi D_i u|$, where we use the fact that $|f'(\cdot)| \le 1$, $|\text{sign}(\cdot)| \le 1$.

Thus, we have established that $D_i|u| = \operatorname{sign}(u) D_i u$, which is clearly in $L^2(\Omega)$.

2 Suppose that Ω is a bounded domain, and $\beta \in C^1(\bar\Omega)$ and $p \in H^1(\Omega)$. Show that $\beta p \in H^1(\Omega)$.

Proof: The proof agin relies on approximating $p \in H^1(\Omega)$ using regular functions $p_k \in H^1(\Omega) \cap C^1(\Omega)$. For any such p_k and $\phi \in C_0^\infty(\Omega)$ we have:

$$\begin{aligned}\int_\Omega \beta(x)p_k(x)D_i\phi(x)\,\mathrm{d}x &= -\int_\Omega D_i[\beta(x)p_k(x)]\phi(x)\,\mathrm{d}x \\ &= -\int_\Omega \{[D_i\beta(x)]p_k(x) + \beta(x)[D_ip_k(x)]\phi(x)\,\mathrm{d}x,\end{aligned}$$

where we used integration by parts and Leibniz product rule for regular functions. Finally one needs to take the limits as $k \to \infty$ on both sides of this equality, exactly as in the previous exercise part b) and c).

Thus we know that the weak derivtive $D_i[\beta p] = [D_i\beta]p + \beta[D_ip]$, and we need to show that it and the function βp are square integrable. This is true because both β and $D_i\beta$ are continuous on the closed bounded set $\bar{\Omega}$ and are therefore bounded on this set.