Exercise set 5

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Please read section 2.15 in [Tr]. Note that the regularity of optimal controls relies upon the preservation of the weak differentiability by the projection map $\mathbb{P}_{[u_a,u_b]}(u) = \max\{u_a, \min\{u_b, u\}\}$. Since $\max\{u_1, u_2\} = (|u_1 - u_2| + u_1 + u_2)/2$, this issue hinges upon the preservation of the weak differentiability by the absolute value map.

$\boxed{1}$ Let Ω be an open set in $\mathbb{R}^N$. We will prove the following fact: if $u \in H^1(\Omega)$ then $|u| \in H^1(\Omega)$.

For $\epsilon > 0$ we will use the following regularization (approximation) of $|\cdot|$: $f_\epsilon(t) = (t^2 + \epsilon^2)^{1/2}$.

a) Let $\epsilon > 0$. Show that f_ϵ is a Lipschitz continuous function with Lipschitz constant 1.

The second "trick" is the density of "nice" functions, for example $C^1(\Omega)$, in $H^1(\Omega)$. For any $u_k \in H^1(\Omega) \cap C^1(\Omega)$ and any $\phi \in C_0^\infty(\Omega)$ we have the equality

$$\int_\Omega f_\epsilon(u_k) D_i\phi = -\int_\Omega \phi f'_\epsilon(u_k) D_i u_k.$$

Furthermore, for any $u \in H^1(\Omega)$ there is a sequence of $u_k \in H^1(\Omega) \cap C^1(\Omega)$ such that that $\lim_{k\to\infty} \|u_k - u\|_{H^1(\Omega)} = 0$. Per definition, it means that both the function values and its derivatives converge in $L^2(\Omega)$, and thus converge almost everywhere pointwise for some subsequence. We relabel $\{u_k\}$ to be this subsequence. We now want to show that both sides of the integral equality above are continuous with respect to this type of convergence.

b) Use the Lipschitz continuity of f_ϵ to show that

$$\lim_{k\to\infty} \int_\Omega |(f_\epsilon(u_k) - f_\epsilon(u)) D_i\phi| = 0.$$

c) Show that

$$\lim_{k\to\infty} \int_\Omega |\phi[f'_\epsilon(u_k) D_i u_k - f'_\epsilon(u) D_i u| = 0.$$

At this point we know that $\forall u \in H^1(\Omega)$, $\phi \in C_0^\infty(\Omega)$ we have the equality

$$\int_\Omega f_\epsilon(u) D_i\phi = -\int_\Omega \phi f'_\epsilon(u) D_i u.$$

We now let $\epsilon \to 0$, show that both sides of the equality converge, and identify the limits.

d) Show that
$$\lim_{\epsilon\to\infty}\int_\Omega |[f_\epsilon(u)-|u|]D_i\phi| = 0.$$

e) Finally, show that
$$\lim_{\epsilon\to 0}\int_\Omega \phi|f_\epsilon'(u)-\operatorname{sign} u|D_iu = 0.$$

Thus, we have established that $D_i|u| = \operatorname{sign}(u)D_iu$, which is clearly in $L^2(\Omega)$. Thus $|u| \in H^1(\Omega)$.

$\boxed{2}$ Suppose that Ω is a bounded domain, and $\beta \in C^1(\bar{\Omega})$ and $p \in H^1(\Omega)$. Show that $\beta p \in H^1(\Omega)$.