TMA4183 Opt. II
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Exercise set 3

Norwegian University of Science and Technology
Department of Mathematical Sciences

Reading material: Sections 2.5–2.7 from [Tröltsch].

1 Exercise 2.11 [Tr]:

a) Show that $f(u) = \sin(u(1))$ is continuously Frechet differentiable in $C[0,1]$ (i.e., it is Frechet differentiable and the derivative is a continuous function).

Solution: We compute the first variation:

$$\delta f(u,h) = \lim_{\varepsilon \downarrow 0} \frac{\sin(u(1)+\varepsilon h(1)) - \sin(u(1))}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}[\sin(u(1)+\varepsilon h(1))]|_{\varepsilon=0} = \cos(u(1))h(1).$$

Since $\delta f(u,h)$ is linear with respect to h and also is bounded:

$$|\delta f(u,h)| \le |\cos(u(1))||h(1)| \le |\cos(u(1))| \sup_{x\in[0,1]} |h(x)| = |\cos(u(1))|\|h\|_{C[0,1]},$$

f is in fact Gateaux differentiable. Furthermore

$$\begin{aligned} &\lim_{\|h\|_{C[0,1]}\to 0} \frac{|\sin(u(1)+h(1)) - \sin(u(1)) - \cos(u(1))h(1)|}{\|h\|_{C[0,1]}} \\ &\le \lim_{\|h\|_{C[0,1]}\to 0} \frac{|-\sin(u(1)+\theta h(1))h^2(1)/2|}{\|h\|_{C[0,1]}} \le \lim_{\|h\|_{C[0,1]}\to 0} \frac{\|h\|^2_{C[0,1]}}{2\|h\|_{C[0,1]}} = 0, \end{aligned}$$

$\theta \in [0,1]$, where we used a second order Taylor series expansion of sin and the fact that $|\sin| \le 1$. Thus f is Frechet differentiable.
Finally we have that

$$\begin{aligned} \|f'(u_1) - f'(u_2)\|_{\mathcal{L}(C[0,1],\mathbb{R})} &\le \sup_{h\neq 0} \frac{|[\cos(u_1(1)) - \cos(u_2(1))]h(1)|}{\|h\|_{C[0,1]}} \\ &\le |\sin(u_2(1) + \theta(u_1(1) - u_2(1)))(u_1(1) - u_2(1))| \\ &\le |u_1(1) - u_2(1)| \le \|u_1 - u_2\|_{C[0,1]}. \end{aligned}$$

Thus f' is continuous (in fact, Lipschitz continuous).

b) Show that $f(u) = \|u\|^2_H$ is continuously Frechet differentiable in H, where H is an arbitrary Hilbert space.

Solution:

We have the equality $f(u+h) - f(u) = 2(u,h) + \|h\|_H^2$, and from which it follows that

$$\lim_{h\to 0} \frac{|f(u+h) - f(u) - 2(u,h)|}{\|h\|_H} = \lim_{h\to 0} \frac{\|h\|_H^2}{\|h\|_H} = 0.$$

and therefore $f'(u)h = 2(u,h)$.
Now

$$\|f'(u_1) - f'(u_2)\|_{H'} = \sup_{h\neq 0} \frac{|2(u_1,h) - 2(u_2,h)|}{\|h\|_H} \le \sup_{h\neq 0} \frac{2\|u_1 - u_2\|_H \|h\|_H}{\|h\|_H} = 2\|u_1 - u_2\|_H,$$

and thus the derivative is a Lipschitz continuous function.

c) $C[0,1]$ is everywhere dense in $L^2(0,1)$. Does this imply that $f(u)$ in a) is continuously Frechet differentiable on $L^2(0,1)$?

Solution:
No, Frechet differentiable functions must be continuous, whereas $u(1)$ is not continuous on $L^2[0,1]$. (E.g. consider the sequence $u_k(x) = x^k \to 0$ in L^2, yet $f(u_k) = 1 \not\to f(0) = 0$.)

2 Let H be a Hilbert space, $a : H \times H \to \mathbb{R}$ be a symmetric, bounded and coercive bilinear form, and $L \in H'$ be a bounded linear functional. Let us define $f : H \to \mathbb{R}$ by $f(u) = a(u,u)/2 - L(u)$. Show that f is continuously Frechet differentiable on H. Express the condition $f' = 0$ as a variational problem.

Solution:

Similarly to the inner product we have the expansion $f(u+h) - f(u) = a(u,h) + a(h,h)/2 - L(h)$ (we have used the symmetry here). Owing to the boundedness of a we immediately get that $f'(u)h = a(u,h) - L(h)$ and that f' is a continuous function. As a result, $f'(u) = 0$ in $H' \iff f'(u)h = 0, \forall h \in H \iff a(u,h) = L(h), \forall h \in H$.

3 * Let H be a Hilbert space, $C \subset H$ be a non-empty closed convex subset of H, and finally $x \in H \setminus C$. Show that x and C can be separated: that is, there is $f \in H'$, $\alpha \in \mathbb{R}$, such that $\forall y \in C$ we have $f(y) \le \alpha$ and $f(x) > \alpha$.

Hint: let $\hat{x}$ be the unique projection of x onto C. Define $f(y) = (x - \hat{x}, y)$. The separation follows from the first order necessary optimality conditions for the projection.

Solution: Let $F(y) = 1/2\|y - x\|_H^2$. Then, by definition, $F(\hat{x}) = \min_{y\in C} F(y)$. Note that because C is closed and $x \in H \setminus C$ we have that $F(\hat{x}) > 0$.

The first order necessary optimality conditions state that $\forall y \in C$ we have the inequality $F'(\hat{x})[y - \hat{x}] = (\hat{x} - x, y - \hat{x})_H \ge 0$, or equivalently $\forall y \in C$: $f(y) \le f(\hat{x}) =: \alpha$. Finally note that $f(x - \hat{x}) = 2F(\hat{x}) > 0$ and therefore $f(x) > f(\hat{x}) = \alpha$.

4 Let Ω be a bounded Lipschitz domain and consider the "identity" operator $i : W^{1,2}(\Omega) \to L^2(\Omega)$, defined as $i(u) = u$. Describe its Hilbert space adjoint i^*, that is, for a given $v \in L^2(\Omega)$ state the variational problem solved by $i^*(v)$. Find the PDE/boundary value problem, whose weak solution is given by $i^*(v)$.

Solution:

For the Hilbert space adjoint we have:

$$(i^*(v), u)_{W^{1,2}(\Omega)} = (v, i(u))_{L^2(\Omega)}, \qquad \forall u \in W^{1,2}(\Omega), v \in L^2(\Omega).$$

Thus we can define a symmetric, continuous, and coercive bilinear form $a(u_1, u_2) = (u_1, u_2)_{W^{1,2}(\Omega)}$ on $[W^{1,2}(\Omega)]^2$ and a linear bounded functional $L_v(u) = (v, u)_{L^2(\Omega)}$ on $W^{1,2}(\Omega)$. Then $i^*(v)$ is the unique solution of the variational problem $a(i^*(v), u) = L_v(u)$, $\forall u \in W^{1,2}(\Omega)$.

Performing integration by parts one can see that

$$a(i^*(v), u) = -\int_\Omega [-\Delta i^*(v) + i^*(v)]u + \int_{\partial\Omega} [n \cdot \nabla i^*(v)]u,$$

and therefore the variational problem is nothing else but the weak formulation of the boundary value problem:

$$\begin{aligned} -\Delta i^*(v) + i^*(v) &= v, && \text{in } \Omega, \\ n \cdot \nabla i^*(v) &= 0, && \text{on } \partial\Omega. \end{aligned}$$

5 Exercise 2.10 [Tr]:
Suppose that Y and U are Hilbert spaces, and let $y_d \in U$, $\lambda \geq 0$, and operator $S \in \mathcal{L}(U, Y)$ be given. Show that the functional

$$f(u) = \|Su - y_d\|_Y^2 + \hat{\lambda}\|u\|_U^2$$

is strictly convex if $\hat{\lambda} > 0$ or S is injective.

Solution:

The functional f is convex.

Take arbitrary $u_1 \neq u_2$, and put $u = (u_1 + u_2)/2$. We will show that $f(u) < (f(u_1) + f(u_2))/2$. Then, since any point on the line segment $[u_1, u_2]$ can be written as a convex combination of either u_1 and u or u and u_2, this will imply strict convexity, e.g., if $0 < \lambda < 1/2$ then

$$\begin{aligned} f(\lambda u_1 + (1-\lambda)u_2) &= f(2\lambda u + (1-2\lambda)u_2) \leq 2\lambda f(u) + (1-2\lambda)f(u_2) \\ &< \lambda(f(u_1) + f(u_2)) + (1-2\lambda)f(u_2) = \lambda f(u_1) + (1-\lambda)f(u_2), \end{aligned}$$

and similarly for $1/2 < \lambda < 1$.

Let us now compute these quantities in our case:

$$\begin{aligned} f(u) &= \|1/2(Su_1 - y_d) + 1/2(Su_2 - y_d)\|_Y^2 + \hat{\lambda}\|1/2u_1 + 1/2u_2\|_U^2 \\ &= 1/4[\|Su_1 - y_d\|_Y^2 + \|Su_2 - y_d\|_Y^2 + \hat{\lambda}\|u_1\|_U^2 + \hat{\lambda}\|u_2\|_U^2] \\ &\quad + 1/2(Su_1 - y_d, Su_2 - y_d)_Y + \hat{\lambda}/2(u_1, u_2)_U, \\ (f(u_1) + f(u_2))/2 &= 1/2[\|Su_1 - y_d\|_Y^2 + \|Su_2 - y_d\|_Y^2 + \hat{\lambda}\|u_1\|_U^2 + \hat{\lambda}\|u_2\|_U^2]. \end{aligned}$$

Therefore

$$(f(u_1) + f(u_2))/2 - f(u) = 1/4\|S(u_1 - u_2)\|_Y^2 + \hat{\lambda}/4\|u_1 - u_2\|_U^2 \geq 0.$$

Note that the equality cannot be attained for $u_1 \neq u_2$ if either $\hat{\lambda} > 0$ or S is injective.