Norwegian University of Science and Technology
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TMA4183 Opt. II
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Exercise set 3

Reading material: Sections 2.5–2.7 from [Tröltsch].

1 Exercise 2.11 [Tr]:

a) Show that $f(u) = \sin(u(1))$ is continuously Frechet differentiable in $C[0, 1]$ (i.e., it is Frechet differentiable and the derivative is a continuous function).

b) Show that $f(u) = \|u\|_H^2$ is continuously Frechet differentiable in H, where H is an arbitrary Hilbert space.

c) $C[0, 1]$ is everywhere dense in $L^2(0, 1)$. Does this imply that $f(u)$ in a) is continuously Frechet differentiable on $L^2(0, 1)$?

2 Let H be a Hilbert space, $a : H \times H \to \mathbb{R}$ be a symmetric, bounded and coercive bilinear form, and $L \in H'$ be a bounded linear functional. Let us define $f : H \to \mathbb{R}$ by $f(u) = a(u, u)/2 - L(u)$. Show that f is continuously Frechet differentiable on H. Express the condition $f' = 0$ as a variational problem.

3 * Let H be a Hilbert space, $C \subset H$ be a non-empty closed convex subset of H, and finally $x \in H \setminus C$. Show that x and C can be separated: that is, there is $f \in H'$, $\alpha \in \mathbb{R}$, such that $\forall y \in C$ we have $f(y) \leq \alpha$ and $f(x) > \alpha$.

Hint: let $\hat{x}$ be the unique projection of x onto C. Define $f(y) = (x - \hat{x}, y)$. The separation follows from the first order necessary optimality conditions for the projection.

4 Let Ω be a bounded Lipschitz domain and consider the "identity" operator $i : W^{1,2}(\Omega) \to L^2(\Omega)$, defined as $i(u) = u$. Describe its Hilbert space adjoint i^*, that is, for a given $v \in L^2(\Omega)$ state the variational problem solved by $i^*(v)$. Find the PDE/boundary value problem, whose weak solution is given by $i^*(v)$.

5 Exercise 2.10 [Tr]:
Suppose that Y and U are Hilbert spaces, and let $y_d \in U$, $\lambda \geq 0$, and operator $S \in \mathcal{L}(U, Y)$ be given. Show that the functional

$$f(u) = \|Su - y_d\|_Y^2 + \hat{\lambda}\|u\|_U^2$$

is strictly convex if $\hat{\lambda} > 0$ or S is injective.

Hint: show "midpoint strict convexity": $f((u_1 + u_2)/2) < [f(u_1) + f(u_2)]/2$. This combied with (non-strict) convexity of f can be used to show strict convexity.