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TMA4183 Opt. II
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Exercise set 2

1 **a)** Show that the weak derivative of $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = |x|$ is

$$g(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Note that it is not necessary to define g at 0, which has measure 0. Thus $f \in W^{1,p}(a, b)$ for an arbitrary $a < b$ and arbitrary $1 \le p \le \infty$.

Solution: Indeed for arbitrary $a < 0 < b$ and an arbitrary $\phi \in C_0^\infty(a, b)$ we have

$$\int_a^b |x|\phi'(x)\,\mathrm{d}x = -\int_a^0 x\phi'(x) + \int_0^b x\phi'(x) = \int_a^0 \phi(x) - \int_0^b \phi(x) = -1\int_a^b g(x)\phi(x)\,\mathrm{d}x,$$

where the second equality is obtained using integration by parts while noting that $\phi(a) = \phi(b) = 0$ and $x|_0 = 0$. Thus g, per definition, is the weak derivative of f.

b) Show that f in the previous example is *not* twice weakly differentiable. (This example shows than not all functions are weakly differentiable.)
Hint: take an arbitrary $\phi \in C_0^\infty(\mathbb{R})$, such that $\phi(0) \neq 0$, and put $\phi_k(x) = \phi(kx)$. Assume that equality (2.1) in the book holds for some integrable function (=potential weak derivative), and consider the limit of both sides of the equality for $k \to \infty$. Use the dominated Lebesgue convergence theorem to switch from the pointwise convergence of ϕ_k to the convergence of the integrals.

Solution: Assume that the weak second derivative of f exists and equals h, that is, for any $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\int f(x)\phi''(x) = \int h(x)\phi(x).$$

Note that if supp $\phi \subset [-N, N]$ then also supp $\phi' \subset [-N, N]$ and in particular $\phi' \in C_0^\infty(\mathbb{R})$. Therefore, owing to (a) we get

$$\int f(x)\phi''(x) = -\int g(x)\phi'(x),$$

thus the weak second derivative of f is the weak first derivative of g.
Let us now assume that $\phi(0) \neq 0$ and construct $\phi_k(x) = \phi(kx)$. Then $\phi_k(0) = \phi(0) \neq 0$ and supp $\phi_k \subset [-N/k, N/k]$. In particular, for any $x \neq 0$ we have $\phi_k(x) = \phi(kx) = 0$ for $k > N/|x|$. Thus $\phi_k(x) \to 0$ as $k \to \infty$, pointwise, almost everywhere (in this case ewerywhere except at $x = 0$).

Finally, we compute

$$-\int g(x)\phi_k'(x) = \int_{N/k}^{0} \phi_k'(x) - \int_0^{N/k} \phi_k'(x) = \phi_k(0) + \phi_k(0) = 2\phi_k(0) = 2\phi(0) \neq 0.$$

On the other hand we know that $|\phi_k(x)h(x)| \leq \|\phi_k\|_{L^\infty(\mathbb{R})}|h(x)| = \|\phi\|_{L^\infty(\mathbb{R})}|h(x)|$, and $|h(x)|$ is a Lebesgue integrable function on $[-N, N]$ (from our assumption of twice weak differentiability of f). Therefore Lebesgue dominated convergence theorem applies and

$$\int_{-N}^{N} h(x)\phi_k(x) \to \int_{-N}^{N} h(x) \cdot 0 = 0 \neq 2\phi_k(0) = -\int_{-N}^{N} g(x)\phi_k'(x)$$

which is a contradiction. Thus f is not two times weakly differentiable.

c) * Cet B be an open unit ball in $\mathbb{R}^n$, and define $f(x) = \|x\|^{-\gamma}$, $\gamma > 0$. Note that the function "blows up" at 0 but is in $C^\infty(B \setminus \{0\})$. Let $g(x) = \nabla f(x)$ for $x \neq 0$. Derive the conditions on γ to show that g is the weak derivative of f in B. This example shows that some discontinuous/unbounded functions are weakly differentiable.

Hint: fix an arbitrary $\phi \in C_0^\infty(B)$. Then derive bounds on γ under which both f and g are integrable in B, and the integrals in red converge to zero as $\varepsilon \to 0$:

$$\int_B fD_i\phi = \int_{B\setminus\varepsilon B} fD_i\phi + \int_{\varepsilon B} fD_i\phi,$$

$$\int_B g_i\phi = \int_{B\setminus\varepsilon B} g_i\phi + \int_{\varepsilon B} g_i\phi,$$

$$\int_{B\setminus\varepsilon B} fD_i\phi + \int_{B\setminus\varepsilon B} g_i\phi = \int_{\partial\varepsilon B} f\phi\nu_i,$$

where ν is the unit normal to $B \setminus \varepsilon B$. Note that the last equation is the classical integration by parts formula, which can be used because both $f, g, \phi \in C^\infty(B \setminus \varepsilon B)$. Use spherical coordinates to estimate the "small" integrals.
Conclude the proof by observing that

$$\int_B fD_i\phi + \int_B g_i\phi \to \int_{B\setminus\varepsilon B} fD_i\phi + \int_{B\setminus\varepsilon B} g_i\phi \to 0$$

Solution:
So the main problem is to remove the singularity at 0, because the function is differentiable elsewhere. Indeed, let $g(x) = \nabla\|x\|^{-\gamma} = -\gamma\|x\|^{-\gamma-1}\nabla\|x\| = -\gamma\|x\|^{-\gamma-2}x$ for $x \neq 0$. In particular $|g_i(x)| \leq \|g(x)\| = \gamma\|x\|^{-\gamma-1}$.
Let us fix an arbitrary $\phi \in C_0^\infty(B)$, and let us estimate the integrals around the singularity. We do this by using hyperspherical coordinates, and by C_n we denote the surface of the unit sphere in $\mathbb{R}^n$.

$$\left|\int_{\varepsilon B} fD_i\phi\right| \leq \|D_i\phi\|_{L^\infty(B)} \int_{\varepsilon B} |f| = \|D_i\phi\|_{L^\infty(B)} C_n \int_0^\varepsilon r^{n-1}r^{-\gamma}$$
$$= \|D_i\phi\|_{L^\infty(B)} C_n \left[\frac{r^{n-\gamma}}{n-\gamma}\right]_{r=0}^{\varepsilon} \to 0$$

as $\varepsilon \to 0$ for all $\gamma < n$.
Similarly

$$\left| \int_{\varepsilon B} g_i \phi \right| \leq \gamma \|\phi\|_{L^\infty(B)} \int_{\varepsilon B} \|x\|^{-\gamma-1} = \gamma \|\phi\|_{L^\infty(B)} C_n \int_0^\varepsilon r^{n-1} r^{-\gamma-1}$$
$$= \|\phi\|_{L^\infty(B)} \gamma C_n \left[\frac{r^{n-\gamma-1}}{n-\gamma-1} \right]_{r=0}^{\varepsilon} \to 0$$

for all $\gamma < n-1$.
For the surface integral we get

$$\left| \int_{\partial \varepsilon B} f \phi \nu_i \right| \leq \|\phi\|_{L^\infty(B)} \int_{\partial \varepsilon B} |f| = \|D_i \phi\|_{L^\infty(B)} C_n \varepsilon^{n-1} \varepsilon^{-\gamma} \to 0$$

for all $\gamma < n-1$.
As a result we can write

$$\begin{aligned} \int_B f D_i \phi + \int_B g_i \phi &= \int_{B \setminus \varepsilon B} f D_i \phi + \int_{B \setminus \varepsilon B} g_i \phi + \int_{\varepsilon B} f D_i \phi + \int_{\varepsilon B} g_i \phi \\ &= \int_{B \setminus \varepsilon B} D_i [f \phi] + \int_{\varepsilon B} f D_i \phi + \int_{\varepsilon B} g_i \phi \\ &= - \int_{\partial \varepsilon B} f \phi \nu_i + \underbrace{\int_{\partial B} f \phi \nu_i}_{=0 \text{ since } \phi \in C_0^\infty(B)} + \int_{\varepsilon B} f D_i \phi + \int_{\varepsilon B} g_i \phi \to 0. \end{aligned}$$

Since the left hand side is independent from ε we must have the equality

$$\int_B f D_i \phi = - \int_B g_i \phi,$$

for any $\phi \in C_0^\infty(B)$, or that g is the weak derivative of f as long as $\gamma < n-1$.
This does not exclude unbounded functions for $n > 1$!
Of course if further regularity is required, for example that both f and g are square integrable, further restrictions on γ arise.

2 Let Ω be an non-empty open set, $1 \leq p \leq \infty$, and let $u_a, u_b \in L^p(\Omega)$ be such that $u_a(x) \leq u_b(x)$, for almost all $x \in \Omega$. Define $U_{\text{adm}} = \{u \in L^p(\Omega) \mid u_a(x) \leq u(x) \leq u_b(x), \text{for almost all } x\}$. Show that U_{adm} is a closed, convex, and bounded subset of $L^p(\Omega)$. (Hint: to prove closedness, use the fact that convergence of functions in $L^p(\Omega)$ implies, up to a subsequence, convergence almost everywhere in Ω. To show boundedness you could e.g. use the fact that $\max\{|u_a|, |u_b|\} = (|u_a| + |u_b| + ||u_a| - |u_b||)/2$.)

Solution:

(i) Convexity of U_{adm}: Suppose that $u_1, u_2 \in U_{\text{adm}}$. Then for almost all $x \in \Omega$: $u_a(x) \leq u_1(x) \leq u_b(x)$ and $u_a(x) \leq u_2(x) \leq u_b(x)$. Let us multiply these inequalities with non-negative coefficients λ and $1-\lambda$ and add them up. We get: $\lambda u_a(x) + (1-\lambda) u_a(x) = u_a(x) \leq \lambda u_1(x) + (1-\lambda) u_2(x) \leq u_b(x)$, for almost all $x \in \Omega$. Thus $\lambda u_1 + (1-\lambda) u_2 \in U_{\text{adm}}$ for all $\lambda \in [0,1]$, and therefore U_{adm} is a convex set.

(ii) Boundedness of U_{adm}: If $u \in U_{\text{adm}}$ then $|u(x)| \leq \max\{|u_a(x)|, |u_b(x)|\}$, for almost all $x \in \Omega$. We know that $|u_a|, |u_b| \in L^p(\Omega)$ (follows from the definition of $L^p(\Omega)$-norm - since we know that $u_a, u_b \in L^p(\Omega)$). Since L^p is a vector space, we have that $|u_a| - |u_b| \in L^2(\Omega)$, and as a result also $||u_a| - |u_b|| \in L^p(\Omega)$. Therefore $|u(x)| \leq g(x)$ for almost all $x \in \Omega$ where $g(x) = |u_a(x)| + |u_b(x)| + ||u_a(x)| - |u_b(x)|| \in L^p(\Omega)$. In particular, $\|u\|_{L^p(\Omega)} \leq \|g\|_{L^p(\Omega)}$, $\forall u \in U_{\text{adm}}$.

(iii) Closedness of U_{adm}: Suppse that $u_k \in U_{\text{adm}}$ are such that $\|u_k - \bar{u}\|_{L^p(\Omega)} \to 0$, for some $\bar{u} \in L^p(\Omega)$. We need to show that $\bar{u} \in U_{\text{adm}}$. For all k and almost all $x \in \Omega$ we have that $u_a(x) \leq u_k(x) \leq u_b(x)$. Furthermore, we can extract a subsequence from u_k, which we denote by $u_{k'}$, such that for almost all $x \in \Omega$: $\lim_{k' \to \infty} u_{k'}(x) = \bar{u}(x)$. (Convergence in $L^p(\Omega)$ implies pointwise convergence, up to a subsequence.) By taking a limit in the inequalities $u_a(x) \leq u_{k'}(x) \leq u_b(x)$ along $k' \to \infty$ for almost all $x \in \Omega$ we establish that $\bar{u} \in U_{\text{adm}}$.

3 Let H be a Hilbert space.

a) Exercise 2.8 [Tr]:
Assume that $H \ni v_n \rightharpoonup v \in H$ and $H \ni u_n \to u \in H$. Show that $(u_n, v_n) \to (u, v)$.

Solution:

$$|(u_n, v_n) - (u, v)| \leq |(u_n - u, v_n)| + |(u, v_n - v)| \leq [\sup_k \|v_k\|_H] \|u_n - u\|_H + (u, v_n - v),$$

where the second inequality is owing to Cauchy–Schwarz. Since v_k converges weakly, it is also bounded (this follows from the uniform boundedness principle (Banach-Steinhaus theorem) mentioned on p. 44 in [Tr].) Therefore the first term must converge to zero because u_n converges strongly to u in H. The last term goes to zero because v_n converges weakly to v, and in particular $(u, v_n) \to (u, v)$.

b) Construct an example where $H \ni u_n \rightharpoonup u \in H$ and $H \ni v_n \rightharpoonup v \in H$, but $(u_n, v_n) \not\to (u, v)$. (Hint: it is sufficient to consider $u_n = v_n$.)

Solution:
Take $u_n = v_n = e_n$ for an orthornormal basis in any Hilbert space (see example (iii) in [Tr], p. 44). Then $u_n \rightharpoonup 0$ but $(u_n, u_n) = \|u_n\|_H^2 = 1 \not\to 0 = (0, 0) = \|0\|_H^2$.

c) Show that if $H \ni u_n \rightharpoonup u \in H$ and in addition $\|u_n\| \to \|u\|$ then also $u_n \to u$.

Solution:

$$\|u_n - u\|_H^2 = \|u_n\|^2 + \|u\|^2 - 2(u, u_n) \to \|u\|^2 + \|u\|^2 - 2(u, u) = 0.$$

4 * Let H be a Hilbert space, $L \in H'$, and $a : H \times H \to \mathbb{R}$ be a bilinear form, which is bounded and coercive. That is, $\exists M > 0, \beta > 0$: $\forall x, y \in H$ we have the

inequalities $|a(x,y)| \le M\|x\|\|y\|$ and $\beta\|x\|^2 \le a(x,x)$. Note that we do not assume the symmetry of a.

We consider the variational problem: find $x \in H$ such that $\forall y \in H$: $a(x,y) = L(y)$.

a) Show that the operator $A : H \to H'$ defined by $(Ax)(y) = a(x,y)$ is linear and bounded.

Solution: Linearity and boundedness of A follows immediately from bilinearity and boundedness of a. Indeed, e.g. boundedness:

$$\|Ax\|_{H'} = \sup_{y\neq 0} \frac{|(Ax)(y)|}{\|y\|} = \sup_{y\neq 0} \frac{|a(x,y)|}{\|y\|} \le M\|x\|.$$

By definition A is then bounded.

b) Show that our variational problem is equivalent to solving the equation $Ax = L$.

Solution:

$$Ax = L \iff \forall y \in H : (Ax)(y) = L(y) \iff \forall y \in H : a(x,y) = L(y)$$

c) Let $R : H \to H'$ be the Riesz map, that is, $(Rx)(y) = (x,y)$. Recall that Riesz representation theorem says that this map is $1:1$ and is an isometry.
Show that our variational problem is equivalent to the equation $R^{-1}(L - Ax) = 0$.

Solution: Since R is an invertible isometry between H and H' (it preserves length of vectors, so in particular $Rz = 0 \iff z = 0$). Therefore

$$Ax = L \iff L - Ax = 0 \iff R^{-1}(L - Ax) = R^{-1}(0) = 0.$$

d) Given some $\omega \neq 0$, define an operator $T : H \to H$ by $Tx = x + \omega R^{-1}(L - Ax)$. Show that our variational problem is equivalent to a fixed-point problem $x = Tx$.

Solution:

$$Tx = x \iff x + \omega R^{-1}(L - Ax) = x \iff \omega R^{-1}(L - Ax) = 0 \iff R^{-1}(L - Ax) = 0,$$

because $\omega \neq 0$ by our assumption.

e) * Show that we can always find $\omega \neq 0$ such that Tx is a *contraction*, that is, there is $0 \le \delta < 1$ such that $\forall x, y \in H$: $\|Tx - Ty\| \le \delta\|x - y\|$.

Solution: Let us find the appropriate conditions on $\omega \neq 0$ making T into a contraction:

$$\begin{aligned}
\|Tx - Ty\|_H^2 &= \|(I - \omega R^{-1}A)(x-y)\|_H^2 \\
&= (x-y, x-y) - 2\omega(R^{-1}A(x-y), x-y) + \omega^2(R^{-1}A(x-y), R^{-1}A(x-y)) \\
&= \|x-y\|_H^2 - 2\omega(A(x-y))(x-y) + \omega^2\|R^{-1}A(x-y)\|_H^2 \\
&= \|x-y\|_H^2 - 2\omega a(x-y, x-y) + \omega^2\|A(x-y)\|_{H'}^2 \\
&\le \|x-y\|_H^2 - 2\omega\beta\|x-y\|_H^2 + \omega^2 M^2\|x-y\|_H^2 \\
&= (1 - 2\omega\beta + \omega^2 M^2)\|x-y\|_H^2.
\end{aligned}$$

If we select ω such that $\omega^2 M^2 - 2\omega\beta < 0$ (i.e. $\omega \in (0, 2\beta/M^2)$) yet $\omega^2 M^2 - 2\omega\beta > -1$ (which is satisfied for all $\omega \approx 0$ since $0^2 M^2 - 2 \cdot 0\beta = 0 > -1$) then the contraction property is satisfied with $0 < \delta^2 = 1 - 2\omega\beta + \omega^2 M^2 < 1$.

f) We now select and fix $\omega \neq 0$ found in the previous part. For an arbitrary $x_0 \in H$ we consider the Richardson's iteration: $x_{k+1} = Tx_k$. Show that the sequence $\{x_k\}$ is Cauchy and thus converges towards the unique fixed point of T.[1]

Solution:
We will first show that the sequence $\{x_k\}$ is Cauchy. Indeed,

$$\begin{aligned}\|x_{k+n} - x_k\|_H &\leq \|x_{k+n} - x_{k+n-1}\|_H + \cdots + \|x_{k+1} - x_k\|_H \\ &\leq (\delta^{n-1} + \cdots + 1)\|x_{k+1} - x_k\|_H \leq (\delta^{n-1} + \cdots + 1)\delta^k \|x_1 - x_0\|_H \\ &\leq \left[\sum_{m=0}^{\infty} \delta^m\right] \delta^k \|x_1 - x_0\|_H = \frac{\delta^k}{1-\delta}\|x_1 - x_0\|_H.\end{aligned}$$

Note that the last term goes to 0 as $k \to \infty$. Let $\varepsilon > 0$ be arbitrary and let N be such that $\delta^k/(1-\delta)\|x_1 - x_0\|_H < \varepsilon$, $\forall k \geq N$. Then $\forall m \geq k \geq N$ it holds that $\|x_m - x_k\|_H < \varepsilon$ (we simpy denote $m = k + n$).
The (Hilbert) space H is complete, therefore $\exists \bar{x} \in H$: $\lim_{k\to\infty} x_k = \bar{x}$. T is continuous (in fact it is Lipschitz continuous as a contraction) and therefore $T\bar{x} = \lim_{k\to\infty} Tx_k = \lim_{k\to\infty} x_{k+1} = \bar{x}$. Thus $\bar{x}$ is a fixed point of T.
Suppose that there are two distinct fixed points: $\bar{x}_1 \neq \bar{x}_2$. Then $\bar{x}_1 - \bar{x}_2 = T\bar{x}_1 - T\bar{x}_2$. However, since T is a contraction $\|\bar{x}_1 - \bar{x}_2\|_H \leq \delta\|T\bar{x}_1 - T\bar{x}_2\|_H = \delta\|\bar{x}_1 - \bar{x}_2\|_H$, which is a contradiction since $0 < \delta < 1$.

[1] This part is the classical Banach fixed point theorem.