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TMA4183 Opt. II
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Exercise set 2

1 **a)** Show that the weak derivative of $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = |x|$ is

$$g(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Note that it is not necessary to define g at 0, which has measure 0. Thus $f \in W^{1,p}(a,b)$ for an arbitrary $a < b$ and arbitrary $1 \le p \le \infty$.

b) Show that f in the previous example is *not* twice weakly differentiable. (This example shows than not all functions are weakly differentiable.)
Hint: take an arbitrary $\phi \in C_0^\infty(\mathbb{R})$, such that $\phi(0) \neq 0$, and put $\phi_k(x) = \phi(kx)$. Assume that equality (2.1) in the book holds for some integrable function (=potential weak derivative), and consider the limit of both sides of the equality for $k \to \infty$. Use the dominated Lebesgue convergence theorem to switch from the pointwise convergence of ϕ_k to the convergence of the integrals.

c) * Cet B be an open unit ball in $\mathbb{R}^n$, and define $f(x) = \|x\|^{-\gamma}$, $\gamma > 0$. Note that the function "blows up" at 0 but is in $C^\infty(B \setminus \{0\})$. Let $g(x) = \nabla f(x)$ for $x \neq 0$. Derive the conditions on γ to show that g is the weak derivative of f in B. This example shows that some discontinuous/unbounded functions are weakly differentiable.
Hint: fix an arbitrary $\phi \in C_0^\infty(B)$. Then derive bounds on γ under which both f and g are integrable in B, and the integrals in red converge to zero as $\varepsilon \to 0$:

$$\int_B f D_i\phi = \int_{B\setminus\varepsilon B} f D_i\phi + \int_{\varepsilon B} f D_i\phi,$$
$$\int_B g_i\phi = \int_{B\setminus\varepsilon B} g_i\phi + \int_{\varepsilon B} g_i\phi,$$
$$\int_{B\setminus\varepsilon B} f D_i\phi + \int_{B\setminus\varepsilon B} g_i\phi = \int_{\partial\varepsilon B} f\phi\nu_i,$$

where ν is the unit normal to $B \setminus \varepsilon B$. Note that the last equation is the classical integration by parts formula, which can be used because both $f, g, \phi \in C^\infty(B \setminus \varepsilon B)$. Use spherical coordinates to estimate the "small" integrals.
Conclude the proof by observing that

$$\int_B f D_i\phi + \int_B g_i\phi \to \int_{B\setminus\varepsilon B} f D_i\phi + \int_{B\setminus\varepsilon B} g_i\phi \to 0$$

2 Let Ω be an non-empty open set, $1 \le p \le \infty$, and let $u_a, u_b \in L^p(\Omega)$ be such that $u_a(x) \le u_b(x)$, for almost all $x \in \Omega$. Define $U_{\text{adm}} = \{u \in L^p(\Omega) \mid u_a(x) \le u(x) \le$

$u_b(x)$, for almost all x }. Show that U_{adm} is a closed, convex, and bounded subset of $L^p(\Omega)$. (Hint: to prove closedness, use the fact that convergence of functions in $L^p(\Omega)$ implies, up to a subsequence, convergence almost everywhere in Ω. To show boundedness you could e.g. use the fact that $\max\{|u_a|, |u_b|\} = (|u_a| + |u_b| + ||u_a| - |u_b||)/2$.)

3 Let H be a Hilbert space.

a) Exercise 2.8 [Tr]:
Assume that $H \ni u_n \rightharpoonup u \in H$ and $H \ni v_n \to v \in H$. Show that $(u_n, v_n) \to (u, v)$.

b) Construct an example where $H \ni u_n \rightharpoonup u \in H$ and $H \ni v_n \rightharpoonup v \in H$, but $(u_n, v_n) \not\to (u, v)$. (Hint: it is sufficient to consider $u_n = v_n$.)

c) Show that if $H \ni u_n \rightharpoonup u \in H$ and in addition $\|u_n\| \to \|u\|$ then also $u_n \to u$.

4 * Let H be a Hilbert space, $L \in H'$, and $a : H \times H \to \mathbb{R}$ be a bilinear form, which is bounded and coercive. That is, $\exists M > 0, \beta > 0$: $\forall x, y \in H$ we have the inequalities $|a(x, y)| \leq M\|x\|\|y\|$ and $\beta\|x\|^2 \leq a(x, x)$. Note that we do not assume the symmetry of a.

We consider the variational problem: find $x \in H$ such that $\forall y \in H$: $a(x, y) = L(y)$.

a) Show that the operator $A : H \to H'$ defined by $(Ax)(y) = a(x, y)$ is linear and bounded.

b) Show that our variational problem is equivalent to solving the equation $Ax = L$.

c) Let $R : H \to H'$ be the Riesz map, that is, $(Rx)(y) = (x, y)$. Recall that Riesz representation theorem says that this map is $1 : 1$ and is an isometry.
Show that our variational problem is equivalent to the equation $R^{-1}(L - Ax) = 0$.

d) Given some $\omega \neq 0$, define an operator $T : H \to H$ by $Tx = x + \omega R^{-1}(L - Ax)$. Show that our variational problem is equivalent to a fixed-point problem $x = Tx$.

e) * Show that we can always find $\omega \neq 0$ such that Tx is a *contraction*, that is, there is $0 \leq \delta < 1$ such that $\forall x, y \in H$: $\|Tx - Ty\| \leq \delta\|x - y\|$.

f) We now select and fix $\omega \neq 0$ found in the previous part. For an arbitrary $x_0 \in H$ we consider the Richardson's iteration: $x_{k+1} = Tx_k$. Show that the sequence $\{x_k\}$ is Cauchy and thus converges towards the unique fixed point of T.[1]

[1] This part is the classical Banach fixed point theorem.