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Exercise set 1

Reading material: Chapter 1 & Section 2.1–2.2 from [Tröltsch].

1 We consider a(n artificial) finite-dimensional optimal control problem for $y \in \mathbb{R}^2$ with a control parameter $u \in \mathbb{R}$.

The state equation is:

$$\begin{aligned} y_1 + y_2 &= u, \\ y_2 &= 2u, \end{aligned} \tag{1}$$

and the const functional is

$$J(y, u) = \frac{1}{2}[(y_1 - 1)^2 + (y_2 - 2)^2] + \frac{\lambda}{2}u^2, \tag{2}$$

where $\lambda > 0$.

a) Derive the explicit expressions for the reduced cost functional and its gradient.

b) Formulate the adjoint problem and compute the reduced gradient with the help of the adjoint state.

c) Assuming $U_{\text{ad}} = \mathbb{R}$ state the first order necessary optimality conditions for this problem.

2 Let V be a normed space and V^* be its dual. Verify the fact that the expression

$$\|f\|_{V^*} = \sup_{x \in V \setminus \{0\}} \frac{|f(x)|}{\|x\|_V}$$

defines a norm on V^*.

3 Let H be a Hilbert space, and consider an arbitrary. Consider an arbitrary $y \in H$. Show that the function $f(x) = (x, y)$ defines a bounded linear functional.

4 Let H be a Hilbert space, and consider an arbitrary $f \in H^* \setminus \{0\}$. Let us define $C = \{\, x \in H \mid f(x) = 1 \,\}$.

a) Show that C is a non-empty closed convex set.

b) Let $\hat{y} \in H$ be an arbitrary vector in C. Show that $C = \hat{y} + \ker f$, where $\ker f = \{\, x \in H \mid f(x) = 0 \,\}$.

c) Let $\bar{y} \in H$ be the shortest vector in C, that is, $\bar{y} = \arg\min_{y \in C} \|y\|_H^2$. Show that $\bar{y} \perp \ker f$, that is, $(\bar{y}, z) = 0$ for all $z \in \ker f$. Hint: consider perturbations of $\bar{y} \pm \delta z$, where $\delta \in \mathbb{R}$ and $z \in \ker f$. Use the optimality of $\bar{y}$.

d) Show that $f(x) = (\tilde{y}, x)$, where $\tilde{y} = \bar{y}/\|\bar{y}\|^2$. Hint: consider two cases: $x \in \ker f$ and $x \notin \ker f$. In the latter case $x/f(x) \in C$, to which the result from **b)** can be applied.

e) Show that $\|f\|_{H^*} = \|\tilde{y}\|_H$.

The last exercise is known as the Riesz representation theorem, which constructs an isometry from H^* into H.