Optimal control of natural convection

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1 Introduction

We will consider the problem of detering the distributed heat source driving a forced convection system towards some optimal state. Namely, we look for admissible controls $u \in U_{\mathrm{adm}} = \{\, \tilde{u} \in L^2(\Omega) \mid u_a(x) \leq \tilde{u}(x) \leq u_b(x), \forall x \in \Omega \,\}$, such that the state $y = (\mathcal{U}, \mathcal{P}, \mathcal{T})$ solving the following non-linear coupled system of PDEs:

$$\left.\begin{aligned} -\mathrm{Pr}\Delta\mathcal{U} + \nabla\mathcal{P} &= -\mathrm{Pr}^2\mathrm{Gr}\mathcal{T}e_g, \\ \nabla\cdot\mathcal{U} &= 0, \\ -\Delta\mathcal{T} + \mathcal{U}\cdot\nabla\mathcal{T} &= \beta u, \end{aligned}\right\} \qquad \text{in } \Omega, \tag{1}$$

with the boundary conditions

$$\left.\begin{aligned} \mathcal{U} &= 0, \\ \partial_\nu\mathcal{T} + \alpha\mathcal{T} &= \alpha\mathcal{T}_0 \end{aligned}\right\} \qquad \text{on } \partial\Omega, \tag{2}$$

minimizes a given objective function.

The left-hand side of the first two partial differential equations in (1) is the so-called Stokes system, which described the steady-state of a slow viscous incompressible liquid with velocity $\mathcal{U}$ and pressure $\mathcal{P}$. The first equation of the Stokes system describes the momentum conservation (more-or-less force equilibrium), whereas the second equation states that the fluid is incompressible.

The last equation in (1) is the advection-diffusion equation for the fluid's temperature $\mathcal{T}$.

The boundary conditions (2) are the no-slip condition at the walls for the fluid velocity, and the Robin boundary condition for the temperature.

Other symbols appearing in (1), (2) are as follows:

- Pr: Prandtl number, describing the ratio of momentum diffusivity to thermal diffusivity;
- Gr: Grashof number, describing the ratio of the buoyancy to viscous force acting on a fluid;
- e_g: unit vector describing the "down" direction of the gravity;
- α: thermal diffusivity of the walls;
- $\mathcal{T}_0$: temperature "outside" of the domain;
- βu: volumetric heat source inside the domain.

Note that the first and the last equations in (1) are non-linearly coupled. Indeed, the temperature $\mathcal{T}$ is convected with the fluid's velocity $\mathcal{U}$. Similarly, the warmer parts of the liquid produce an upwards directed buoyant force proportional to $-\mathcal{T}e_g$ acting on the liquid.

Our goal will be to minimize the following functional:

$$J(y,u) = \frac{1}{2}\sum_i \int_\Omega \gamma_i |\mathcal{U}_i| \mathcal{U}_i + \frac{\lambda}{2}\int_\Omega u^2, \tag{3}$$

where the last integral accounts for the cost of heating the domain, whereas the first part computes the weighted (with some given weights γ_i) and "signed" (< 0 if $\mathcal{U}_i < 0$, > 0 if $\mathcal{U}_i > 0$) kinetic energy generated by the fluid. One can imagine that this kinetic energy can be converted to some form of useful work, e.g. to drive a turbine (we will not model this part in the project).

In my experiments I have used the following parameters: $\Omega \in \mathbb{R}^2$ is the rectangle spanned by the points $(0,0)$ and (W,H), $W = 1.0$, $H = 2.0$; $e_g = (-1,0)$, $\mathcal{T}_0 = 1.0$, $\alpha = 1.0$, $\mathrm{Pr} = 0.1$, $\mathrm{Gr} = 10^5$, $\beta = 20$, $\lambda = 10$, $u_a = 0$, $u_b = 1$, $\gamma_1 = 0$, and

$$\gamma_0(x_0,x_1) = \begin{cases} 1, & x_1 >= 0.75H, \\ 0, & \text{otherwise.} \end{cases}$$

but you could try to vary them to get a physical/numerical intuition about the problem.

2 Function spaces

We will consider U_{adm} as a closed convex subset of $L^2(\Omega)$. Further, we will seek the states in the Hilbert space $Y = \{\mathcal{U} \in [W^{1,2}(\Omega)]^2 \mid \mathcal{U}_i|_{\partial\Omega} = 0\} \times L^2(\Omega) \times W^{1,2}(\Omega)$, where the inner product in the product space Y is defined as the sum of the individual inner products.

Note that only the gradient of the pressure $\mathcal{P}$ enters the problem (1), and therefore the pressure is only defined up to an additive constant. Numerically, we will deal with this issue by fixing the pressure $\mathcal{P}$ to be 0 at one point on the boundary $\partial\Omega$.

3 Weak formulation of the state problem

Question 1. Using integration by parts, show that every "classical" solution $y = (\mathcal{U},\mathcal{P},\mathcal{T})$ to (1), (2) satisfies the variational problem $F(y,u) = 0$ in Y', where for every $v = (\mathcal{V},\mathcal{Q},\mathcal{S}) \in Y$ we have

$$\begin{aligned} F(y,u)v = &- \mathrm{Pr}\int_\Omega \nabla\mathcal{U} : \nabla\mathcal{V} + \int_\Omega \mathcal{P}\nabla\cdot\mathcal{V} - \mathrm{Pr}^2\mathrm{Gr}\int_\Omega \mathcal{T} e_g \cdot \mathcal{V} \\ &- \int_\Omega \mathcal{Q}\nabla\cdot\mathcal{U} \\ &- \int_\Omega \nabla\mathcal{T}\cdot\nabla\mathcal{S} + \int_\Omega \mathcal{T}\mathcal{U}\cdot\nabla\mathcal{S} - \int_{\partial\Omega}\alpha(\mathcal{T}-\mathcal{T}_0)\mathcal{S} + \int_\Omega \beta u\mathcal{S}. \end{aligned} \tag{4}$$

Note: one also needs to show that $F(y,u)$ in (4) defines a bounded linear functional on Y. For the non-linear term $\int_\Omega \mathcal{T}\mathcal{U}\cdot\nabla\mathcal{S}$ this can be shown using the continuous embedding of $W^{1,2}(\Omega)$ into $L^4(\Omega)$ (see [Theorem 7.2 in Tröltzsch]) and then applying (generalized) Hölder's inequality to estimate this integral.

See the Python script `convection.py` for an example of how this variational problem can be solved using FENICs

4 A priori estimates

Question 2. Assume that $y = (\mathcal{U}, \mathcal{P}, \mathcal{T}) \in Y$ satisfies (4) for some $u \in U_{\mathrm{adm}}$.

1. Show that $\|\mathcal{U}\|_{[W^{1,2}(\Omega)]^2} \leq C_1 \|\mathcal{T}\|_{L^2(\Omega)}$, where C_1 is independent from y or u.

2. Show that $\|\mathcal{T}\|_{W^{1,2}(\Omega)} \leq C_2(\|\alpha \mathcal{T}_0\|_{L^2(\partial\Omega)} + \|\beta u\|_{L^2(\Omega)})$, , where C_1 is independent from y or u. Hint: see [Theorem 2.6 in Tröltzsch]; show that $\int_\Omega \mathcal{S}\mathcal{U} \cdot \nabla \mathcal{T} = -\int_\Omega \mathcal{T}\mathcal{U} \cdot \nabla \mathcal{S}$.

The bound on the pressure component $\mathcal{P} \leq C_3 \|\mathcal{T}\|_{L^2(\Omega)}$ follows from the standard theory of mixed variational problems, see for example [Lemma 12.2.12 in Brenner and Scott: The Mathematical Theory of Finite Element Methods].

Combine these bounds to conclude that all solutions to (4), if exists, are uniformly bounded in Y for all $u \in U_{\mathrm{adm}}$.

5 Existence of optimal controls assuming existence of solutions to (4)

Question 3. Assume that for every $u \in U_{\mathrm{adm}}$ there is at least one state $y \in Y$ solving (4). Show that under this assumption there is an optimal solution to the control problem:

$$\begin{aligned} \min_{(u,y) \in U_{\mathrm{adm}} \times Y} \quad & J(y,u), \\ \text{s.t.} \quad & y \text{ solves } (4), \end{aligned} \tag{5}$$

where J is given by (3).

Hint: Use the a priori estimates established in the previous section to select a weakly converging subsequence from the minimizing sequence. To deal with some non-linear/non-convex terms involving $\mathcal{U}$, $\mathcal{T}$ one needs compact embedding of $W^{1,2}(\Omega)$ into $L^p(\Omega)$, which is true for all $1 \leq p < +\infty$ in the 2-dimensional situation, and all $1 \leq p < 6$ in the dimension 3 (see [Evans, L.: Partial Differential Equations, section 5.8.1]).

6 Application of the formal Lagrange method

Question 4.
1. Apply the formal Lagrange method to the problem (5). Find the adjoint problem and state the first order necessary optimality conditions for this problem in terms of the state and the adjoint state.

2. Test numerically the directional derivatives of the reduced objective function computed using the adjoint method (this is of course questionable without knowing the existence/uniqueness of solutions to (4) or the continuity/differentiability of the control-to-state operator) against the finite difference approximation.

7 Application of the projected gradient algorithm

Question 5. Implement and test the projected gradient algorithm for solving the problem (5).

Hint: you can stop the iterations when $\|u_k - \Pi_{U_{\mathrm{adm}}}(u_k - \nabla f(u_k))\|_{L^2(\Omega)}$ is "small", where f is the reduced objective function. Use backtracking linesearch and accept the step τ when $f(\Pi_{U_{\mathrm{adm}}}(u_k - \tau \nabla f(u_k))) \leq f(u_k) + cf'(u_k)[\Pi_{U_{\mathrm{adm}}}(u_k - \tau \nabla f(u_k)) - u_k]$, where $0 < c < 1/2$ is a constant; I used $c = 0.1$ in my simulations.

8 Report

In the report, provide brief answers and explanations to the posed questions. Do not include the whole Python code in the report, instead submit commented code separately. You can of course include code snippets in the report if they help to tell the story.

9 Existence of solutions to (4) (sketch*)

Answer the questions in this section only if you have time remaining after completing the previous sections.

The theory of monotone non-linear operators is not applicable to problem (4). However, we can use the following argumentation.

First, we eliminate the pressure variable from the weak formulation (as mentioned previously, existence/uniqueness of pressures follows from the standard theory of mixed variational formulations). Let $Z = \{\mathcal{U} \in W^{1,2}(\Omega) \mid \nabla \cdot \mathcal{U} = 0, \mathcal{U}|_{\partial\Omega} = 0\} \times W^{1,2}(\Omega)$. For $z = (\mathcal{U}, \mathcal{T}) \in Z$ the problem (4) reduces to: $G(y, u) = 0$ in Z', where for every $w = (\mathcal{V}, \mathcal{Q}) \in Z$ we have

$$\begin{aligned} G(z,u)w = &- \mathrm{Pr}\int_\Omega \nabla\mathcal{U} : \nabla\mathcal{V} - \mathrm{Pr}^2\mathrm{Gr}\int_\Omega \mathcal{T}e_g \cdot \mathcal{V} \\ &- \int_\Omega \nabla\mathcal{T}\cdot\nabla\mathcal{S} + \int_\Omega \mathcal{T}\mathcal{U}\cdot\nabla\mathcal{S} - \int_{\partial\Omega}\alpha(\mathcal{T}-\mathcal{T}_0)\mathcal{S} + \int_\Omega \beta u\mathcal{S}. \end{aligned} \tag{6}$$

Let us define the bilinear form $a : Z \times Z \to \mathbb{R}$ by

$$a(z, w) = \mathrm{Pr}\int_\Omega \nabla\mathcal{U} : \nabla\mathcal{V} + \int_\Omega \nabla\mathcal{T}\cdot\nabla\mathcal{S} + \int_{\partial\Omega}\alpha\mathcal{T}\mathcal{S}, \tag{7}$$

and an associated operator $A : Z \to Z'$ by $(Az)(w) = a(z, w)$. From the theory of linear elliptic operators, [Chapter 2, Tröltzsch] it follows that A admits a bounded inverse $A^{-1} : Z' \to Z$.

Furthermore, Z is compactly embedded into $[L^4(\Omega)]^2 \times L^4(\Omega)$ as mentioned previously; let us call this compact embedding operator by i.

Finally, let us define a non-linear operator $H : [L^4(\Omega)]^2 \times L^4(\Omega) \to Z'$ by

$$H(\mathcal{U},\mathcal{T})[\mathcal{V},\mathcal{S}] = -\mathrm{Pr}^2\mathrm{Gr}\int_\Omega \mathcal{T}e_g\cdot\mathcal{V} + \int_{\partial\Omega}\alpha\mathcal{T}_0\mathcal{S} + \int_\Omega \mathcal{T}\mathcal{U}\cdot\nabla\mathcal{S}$$

The problem (6) can be reformulated as a fixed-point problem: find $z = (\mathcal{U}, \mathcal{T}) \in [L^4(\Omega)]^2 \times L^4(\Omega)$ such that $z = i(A^{-1}(H(z)))$.

Question 6. Show that H is bounded and continuous (note that it is not linear, so the two concepts are not the same).

Thus the operator $i \circ A^{-1} \circ H$ is continuous and compact (since all involved operators are bounded/continuous, and the last one is compact).

Question 7. Suppose that for some $0 \le \sigma \le 1$ the point $z \in [L^4(\Omega)]^2 \times L^4(\Omega)$ satisfies the equation $z = \sigma i(A^{-1}(H(z)))$. Show that $\|z\|_{[L^4(\Omega)]^2\times L^4(\Omega)} \le c$, where $c > 0$ is independent from z and σ.

Hint: arguments are very similar to a priori estimates!

The positive answer to the previous question is sufficient for asserting that the fixed point problem $z = i(A^{-1}(H(z)))$ admits at least one solution, according to Leray–Schauder theorem (see for example [E. Zeidler, Nonlinear Functional Analysis and its Applications]). Note that the uniqueness of solutions is not claimed!