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TMA4183 Opt. II
Spring 2016

Exercise set Project

Model problem

We consider the following model control problem of a semi-linear elliptic PDE:

$$\begin{aligned} \underset{(u,y)}{\text{minimize}} \quad & J(y,u), \\ \text{subject to} \quad & u \in U_{\text{ad}}, \\ & -\Delta y + ud(y) = f, \quad \text{in } \Omega, \\ & y = 0, \qquad\qquad\quad\; \text{on } \Gamma. \end{aligned} \tag{1}$$

where

- Ω is a Lipschitz bounded domain in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$,
- $U_{\text{ad}} = \{\, u \in L^2(\Omega) \mid u_a(x) \leq u(x) \leq u_b(x), \text{a.e. in } \Omega \,\}$ for some given functions $u_a, u_b \in L^\infty(\Omega)$ such that $\hat{\delta} \leq u_a(x) \leq u_b(x)$ for some constant $\hat{\delta} > 0$[1] and almost all $x \in \Omega$,
- the right hand side $f \in L^2(\Omega)$,
- the non-linearity $d(y) = My/(\hat{\epsilon}^2 + y^2)^{1/2}$, where $M > 0$ and $\hat{\epsilon} > 0$ are given constants,
- and finally $J(y,u) = \int_\Omega \phi(x, y(x))\,\mathrm{d}x + \int_\Omega \psi(x, u(x))\,\mathrm{d}x$. The functions ϕ and ψ are assumed to satisfy the assumptions 4.14 (ii) described on p. 206 in [Tr].

The explanation for this model is as follows. The Laplace operator $-\Delta y$ models a steady-state diffusion process, such as for example steady-state temperature or concentration. The non-linearity $d(y)$ approaches the (weak) derivative of $M|y|$ as $\hat{\epsilon} \to 0$, effectively modifying the "heat source" f to be $\approx f - uM$ when $y > 0$ and $\approx f + uM$ when $y < 0$. In turn, if uM is large and positive this enforces the condition $y \approx 0$. As a result, for "bang-bang" type controls $u(x) \in \{u_a(x), u_b(x)\}$ where $u_a \approx 0$, $u_b \approx 1$ we will enforce the condition $y \approx 0$ in the regions where $u = u_b$, whereas the diffusion will be nearly unaffected in the regions where $u = u_a$.

[1] The reason for assuming $\hat{\delta} > 0$ is that we need the control-to-state operator to be differentiable in an open set including U_{ad}. Proving this for $\hat{\delta} = 0$ is somewhat more difficult.

Parameters for numerical experiments

Throughout the project we will assume the following: $\Omega = (0,1)^2$, $u_a = 1.0 \cdot 10^{-4}$, $u_b = 1.0$, $\hat{\epsilon} = 1.0 \cdot 10^{-1}$, $M = 1.0$, $\phi(x, y(x)) = (y(x) - y_d(x))^2/2$ for a suitable function $y_d \in L^2(\Omega)$, $\psi(x, u(x)) = \lambda u(x)/2$ for some constant $\lambda > 0$. Thus it only remains to specify f, y_d, and λ to arrive at a specific instance of (1).

If you want to experiment, you could try using smaller values of $\hat{\epsilon}$ and larger values of M. This makes the problem more difficult to solve numerically.

Project steps

1 Write down the weak form of the state PDE in (1).

2 Sketch a proof of existence and uniqueness of solutions $y \in H^1(\Omega) \cap C(\bar{\Omega})$ for every admissible control $u \in U_{\text{ad}}$ (for example, see exercise 1 from set 7). Furthermore, show that there is a uniform bound on the norm of the states for all admissible controls: $\exists C > 0 : \forall u \in U_{\text{ad}}, \|y\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$.

3 Consider a sequence of controls $u_k \in U_{\text{ad}}$ and a sequence of functions $y_k \in L^2(\Omega)$. Assume that $u_k \rightharpoonup \bar{u}$, weakly in $L^2(\Omega)$, and $y_k \to \bar{y}$, strongly in $L^2(\Omega)$. Show that $u_k d(y_k) \rightharpoonup \bar{u} d(\bar{y})$, weakly in $L^2(\Omega)$.

Hint: see the paragraph immediately after the proof of Lemma 4.11, p. 198 in [Tr] and Exercise 5 a) from set 3.

4 Outline the necessary changes in the proof of Theorem 4.15 (p. 208, [Tr]) to establish the existence of optimal controls to problem (1).

Hint: use the result established in part 3. See exercise 3 in set 8 for "inspiration".

5 Sketch the proof of Frechet differentiability of the control-to-state operator $G : L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega})$.

Hint: for a pair of controls $u_1, u_2 \in U_{\text{ad}}$ and the corresponding pair of states $y_1, y_2 \in H^1(\Omega) \cap C(\bar{\Omega})$, the difference $y_1 - y_2$ solves the system

$$-\Delta(y_1 - y_2) + u_2[d(y_1) - d(y_2)] = -(u_1 - u_2)d(y_1).$$

Let us now consider u_1 to be a perturbation of u_2. Then the right hand side will be small, because $d(\cdot)$ is uniformly bounded. See Section 4.5 in [Tr] for the details.

6 Use the formal Lagrange method to derive the optimality conditions and the adjoint equation. Find the expression for the gradient of the reduced cost function in terms of the state and the adjoint state.

7 Assuming that $\partial_y \phi(\cdot, y(\cdot)) \in L^2(\Omega)$, argue that the adjoint system admits a unique solution in $H^1(\Omega)$ for any $u \in U_{\text{ad}}$.

8 Implement a numerical procedure for solving the state PDE in (1), given $u \in U_{\text{ad}}$ and $f \in L^2(\Omega)$. Verify its convergence on a sequence of refined grids using a method of manufactured solutions (i.e., select $y \in H^1(\Omega)$ satisfying the boundary conditions, and based on this information compute the right hand side f).

Note that the state equation is non-linear, and therefore an iterative procedure is required. Furthermore, the state equations will have to be solved repeatdly, therefore a quickly locally convergent method would be helpful. I recommend implementing Newton's method (with damped steps, if necessary).

9 Finally, implement a projected gradient algorithm with backtracking linesearch for solving the control problem (1).

Try to come up with a test case satisfying the first order (necessary) optimality conditions for verifying the correctness of your implementation: see exercise set 5^{num} for an example.