Cell Centered Finite Volume Discretization of Laplace Equation

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1 Model problem

We consider the following problem: find $y \in H_0^1(\Omega)$ such that

$$-\mathrm{div}[\nabla y] = f, \tag{1}$$

$f \in L^2(\Omega)$. It will be more convenient to characterize y as the unique function in $H_0^1(\Omega)$ satisfying the following variational problem:

$$a(y, z) = \ell(z), \qquad \forall z \in H_0^1(\Omega), \tag{2}$$

where $a : [H_0^1(\Omega)]^2 \to \mathbb{R}$ is a continuous and coercive symmetric bilinear form and $\ell \in [H_0^1(\Omega)]^*$ is a linear continuous functional defined via

$$a(y, z) = \int_\Omega \nabla y(x) \cdot \nabla z(x)\, \mathrm{d}x, \qquad \ell(z) = \int_\Omega f(x) z(x)\, \mathrm{d}x. \tag{3}$$

2 Cell-centered finite volume approximations

2.1 Admissible finite volume discretizations

We utilize finite volume method for discretizing the problem (1). We employ the notation and the approach of [1, 2], which we introduce in this section to keep the paper self-contained.

An admissible finite volume discretization $\mathcal{D}$ of Ω, in the sense of [1], also see Figure 1, is a triple $(\mathcal{T}, \mathcal{E}, \mathcal{P})$, where

(i) $\mathcal{T}$ is a finite collection of non-empty open disjoint convex polygonal subsets of Ω (also known as the "control volumes") such that $\bar{\Omega} = \cup_{K\in\mathcal{T}} \bar{K}$. For each $K \in \mathcal{T}$, ∂K denotes the boundary of K and $\mathrm{m}(K)$ is the d-dimensional Lebesgue measure of K.

(ii) $\mathcal{E}$ is a finite family of disjoint subsets of $\bar{\Omega}$ (the edges in 2D, or (hyper-)faces of the mesh in $d \geq 3$). For every $\sigma \in \mathcal{E}$, there is a $d-1$-dimensional hyperplane $E \subset \mathbb{R}^d$ and $K \in \mathcal{T}$ such that $\bar{\sigma} = \partial K \cap E$ and σ is a non-empty open subset of E. $\mathrm{m}(\sigma)$ will denote the $d-1$-dimensional Lebesgue measure of σ. For every $K \in \mathcal{T}$, there is a subset $\mathcal{E}_K \subseteq \mathcal{E}$ such that $\partial K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$. As a result of these assumptions, every $\sigma \in \mathcal{E}$ is either a subset of $\partial\Omega$, or a face common to two control volumes $K, L \in \mathcal{T}$: $\bar{\sigma} = \bar{K} \cap \bar{L}$. In the latter case we will denote the edge with $K \mid L$. The set of boundary edges will be denoted with $\mathcal{E}_{\text{ext}}$ and the set of interior edges is $\mathcal{E}_{\text{int}}$. For every $\sigma \in \mathcal{E}$ we define its center:

$$x_\sigma = \frac{1}{\mathrm{m}(\sigma)} \int_\sigma x \, \mathrm{d}x.$$

(iii) $\mathcal{P} = \{ x_K \mid K \in \mathcal{T} \}$ is a finite collection of points in Ω, such that for every $K \in \mathcal{T}$ it holds that $x_K \in K$. For every $K \mid L \in \mathcal{E}_{\text{int}}$ we assume that $x_K - x_L$ is orthogonal to $K \mid L$. Finally, let $z_{K,\sigma}$ be an orthogonal projection of x_K onto $\sigma \in \mathcal{E}_K$, then $z_{K,\sigma} \in \sigma$ for every $\sigma \subseteq \partial\Omega$. We denote the distance between x_K and $z_{K,\sigma}$ with $d_{K,\sigma}$, and the distance between x_K and x_L with $d_{K|L}$.

For every discretization $\mathcal{D}$ we measure its size with

$$h_{\mathcal{D}} = \sup_{K \in \mathcal{T}} \operatorname{diam}(K),$$

and its regularity with

$$\theta_{\mathcal{D}} = \inf_{K \in \mathcal{T}, \sigma \in \mathcal{E}_K} \frac{d_{K,\sigma}}{\operatorname{diam}(K)}.$$

For every $K \in \mathcal{T}$ we denote by $\mathcal{N}(K) \subset \mathcal{T}$ the control volumes having a common edge with K (not including K itself). For all $K \mid L \in \mathcal{E}_{\text{int}}$ we denote $\mathbf{n}_{K|L} = d_{K|L}^{-1}(x_K - x_L)$ the unit normal to $K \mid L$ directed from L to K. For every $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$, we write $\mathbf{n}_{K,\sigma} = d_{K,\sigma}^{-1}(z_{K,\sigma} - x_K)$ for a unit outward normal for K on σ. We also set

$$\tau_{K|L} = \frac{\mathrm{m}(K \mid L)}{d_{K|L}}.$$

Similarly, for every $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we set

$$\tau_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{d_{K,\sigma}}.$$

For $\sigma \in \mathcal{E}_{\text{ext}}$, let $K \in \mathcal{T}$ be such that $\sigma \in \mathcal{E}_K$; then we set $\tau_\sigma = \tau_{K,\sigma}$.

2.2 Cell centered approximation of (1)

Given an admissible discretization $\mathcal{D} = (\mathcal{T}, \mathcal{E}, \mathcal{P})$ we are ready to construct approximations of the direct problem (1). Let $H_{\mathcal{D}}(\Omega)$ be a set of functions from Ω to $\mathbb{R}$ constant on every control volume $K \in \mathcal{T}$; this is going to be an approximation space for solutions to the boundary value problem (1).

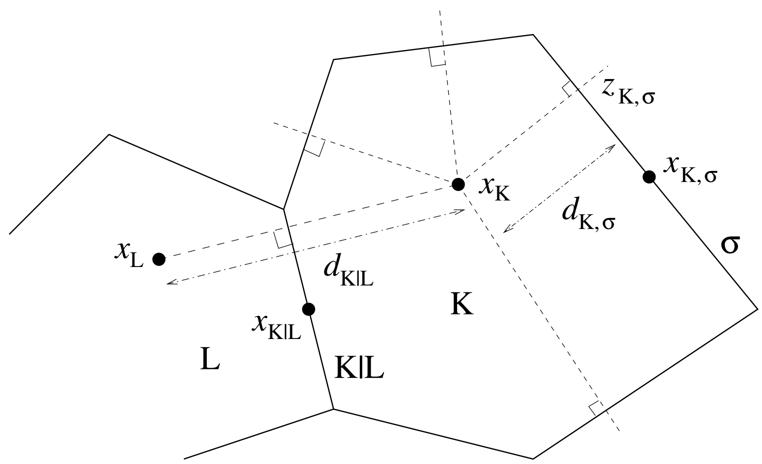

Figure 1: Admissible finite volume discretization with associated notation for a control volume in two dimensions.

After integrating the equation (1) on every control volume $K \in \mathcal{T}$ and utilizing Gauss–Ostrogradsky theorem, we end up with

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(y) = \mathrm{m}(K) f_K, \qquad \forall K \in \mathcal{T}, \tag{4}$$

where $f_K = \mathrm{m}(K)^{-1} \int_K f(x)\,\mathrm{d}x$, and $F_{K,\sigma} : H_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega) \to \mathbb{R}$ is an approximation of the normal flux $\nabla y \cdot \mathbf{n}$ through the edge σ. It is common to use the following central difference approximation of the diffusive fluxes:

$$F_{K,\sigma}(y) = \begin{cases} -\tau_{K|L}(y_L - y_K), & \text{if } \sigma = K \mid L \in \mathcal{E}_{\mathrm{int}}, \\ -\tau_\sigma(-y_K), & \text{if } \sigma \in \mathcal{E}_{\mathrm{ext}}. \end{cases} \tag{5}$$

While the equations (4), (5) provide a complete description of the finite-volume method for (1), for the purpose of the convergence analysis it is more convenient to rewrite (4) in the variational form, see [1, 2]. We introduce a bilinear form $a_{\mathcal{D}} : [H_{\mathcal{D}}(\Omega)]^2 \to \mathbb{R}$ and a linear functional $\ell_{\mathcal{D}} \in [H_{\mathcal{D}}(\Omega)]^*$:

$$\begin{aligned} a_{\mathcal{D}}(y,z) &= \sum_{K|L \in \mathcal{E}_{\mathrm{int}}} \alpha_{K|L} \tau_{K|L} (y_L - y_K)(z_L - z_K) + \sum_{\sigma \in \mathcal{E}_{\mathrm{ext}}} \alpha_\sigma \tau_\sigma y_K z_K, \\ \ell_{\mathcal{D}}(z) &= \sum_{K \in \mathcal{T}} \mathrm{m}(K) f_K z_K = \ell(z). \end{aligned} \tag{6}$$

Using this notation, the flux formulation (4) is equivalent to searching for $y \in H_{\mathcal{D}}(\Omega)$ such that we have the equality:

$$a_{\mathcal{D}}(y,z) = \ell(z), \qquad \forall z \in H_{\mathcal{D}}(\Omega). \tag{7}$$

Note that in a rather rigorous sense the discretized bilinear form $a_{\mathcal{D}}$ is an approximation of the continuous bilinear form a.

References

[1] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In P. G. Ciarlet and J. L. Lions, editors, *Handbook of Numerical Analysis*, volume 7, pages 713–1020. North Holland, 2000.

[2] R. Eymard, T. Gallouët, R. Herbin, and J.-C. Latche. Analysis tools for finite volume schemes. *Acta Math. Univ. Comenianae*, LXXVI(1):111–136, 2007.