TMA4183 Opt. II
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Exercise set 8

Norwegian University of Science and Technology
Department of Mathematical Sciences

Please read sections 4.3–4.4 in [Tr].

1 Exercise 4.4 (ii) in [Tr]: Show that Nemytskii operator $y(\cdot) \mapsto \sin(y(\cdot))$ is Frechet differentiable from $L^{p_1}(0,T)$ into $L^{p_2}(0,T)$ whenever $1 \le p_2 < p_1 \le \infty$.

2 Compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ (Rellich–Kondrachov Theorem, Theorem 7.4 in [Tr]) plays an important role in the proof of Theorem 4.15 (existence of optimal controls for semi-linear elliptic PDEs). There are many other examples of compact embeddings.

Let $-\infty < a < b < +\infty$, and consider the spaces of continuous functions $C^0[a,b]$ and Hölder continuous functions $C^{0,\gamma}[a,b]$, $0 < \gamma \le 1$. These spaces are equipped with the norms

$$\|f\|_{C^0[a,b]} = \sup_{x\in[0,T]} |f(x)|,$$
$$\|f\|_{C^{0,\gamma}[a,b]} = \|f\|_{C^0[a,b]} + \sup_{x\neq y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.$$

We will use Arzela–Ascoli characterization of relative compactness in $C^0[a,b]$ (it is not difficult to prove): The set $S \subset C^0[a,b]$ is relatively compact if and only if it is *bounded* and *equicontinuous*. That is, there is $M > 0$ such that $\forall f \in S : \|f\|_{C^0[a,b]} \le M$, and for every $\epsilon > 0$ there is $\delta > 0$: $\forall f \in S, x, y \in [a,b] : |x-y| < \delta \implies |f(x)-f(y)| < \epsilon$.

a) Show that every bounded subset in $C^{0,\gamma}[a,b]$ is bounded and equicontinuous in $C^0[a,b]$. Conclude that from any bounded sequence in $C^{0,\gamma}[a,b]$ one can extract a convergent sequence in $C^0[a,b]$.

Solution: Assume that $S \subset C^{0,\gamma}[a,b]$ is such that $\exists M > 0 : \forall f \in S, \|f\|_{C^{0,\gamma}[a,b]} \le M$. By definition $\|f\|_{C^0[a,b]} \le \|f\|_{C^{0,\gamma}[a,b]} \le M$ and thus S is also a bounded set in $C^0[a,b]$. Furthermore from the definition of the norm we have that $|f(x)-f(y)| \le |x-y|^\gamma \|f\|_{C^{0,\gamma}[a,b]}$. Thus as long as $|x-y| < \delta$ it follows that $\forall f \in S : |f(x)-f(y)| < \delta^\gamma M$. Thus is is sufficient to choose $\delta = (\varepsilon/M)^{1/\gamma}$ in the definition of equicontinuity.

b) Show that any sequence $f_n \in C^{0,\gamma}[a,b]$, which converges weakly to some limit $\bar{f} \in C^{0,\gamma}[a,b]$, must satisfy $\|f_n - \bar{f}\|_{C^0[a,b]} \to 0$.

Hint: show the inclusion $(C^0[a,b])' \subset (C^{0,\gamma}[a,b])'$ for the dual spaces; use the fact that weakly convergent sequences are bounded (this is known as the uniform boundedness principle in functional analysis); then use the proof by contradiction and **a)**.

Solution: Suppose that $f_n \rightharpoonup \bar{f} \in C^{0,\gamma}[a,b]$. Weakly convergent sequences are bounded (uniform boundedness principle), and thus the set $\{f_n\}$ is relatively compact in $C^0[a,b]$ according to **a)** and Arzela-Ascolli theorem.
Let us now take $F \in (C^0[a,b])'$. Since $C^{0,\gamma}[a,b] \subset C^0[a,b]$ the function F is defined and linear on $C^{0,\gamma}[a,b]$; furthermore $|F(f)| \leq \|F\|_{(C^0[a,b])'}\|f\|_{C^0[a,b]} \leq \|F\|_{(C^0[a,b])'}\|f\|_{C^{0,\gamma}[a,b]}$, $\forall f \in C^{0,\gamma}[a,b]$. As a result, $F \in (C^{0,\gamma}[a,b])'$ and $(C^0[a,b])' \subset (C^{0,\gamma}[a,b])'$. Since we know that $\forall F \in (C^{0,\gamma}[a,b])' : F(f_n) \to F(\bar{f})$, therefore this happens for all $F \in (C^0[a,b])'$ and $f_n \rightharpoonup \bar{f}$ in $C^0[a,b]$ as well.
Finally, assume that $\|f_n - \bar{f}\|_{C^0[a,b]} \not\to 0$, that is, for some $\epsilon > 0$ there is a subsequence n' of n such that $\|f_{n'} - \bar{f}\|_{C^0[a,b]} \geq \epsilon$. Since $\{f_{n'}\}$ is a subset of $\{f_n\}$, a relatively compact set in $C^0[a,b]$, we can extract a further subsequence n'' from it, such that $\|f_{n''} - \tilde{f}\|_{C^0[a,b]} \to 0$, for some $\tilde{f} \in C^0[a,b]$. Owing to the assumptions on n', we have $\tilde{f} \neq \bar{f}$. Thus the subsequence $f_{n''}$ has two weak limits: $\tilde{f}$ (strong convergence implies weak) and $\bar{f}$ (as a subsequence of a weakly convergent sequence f_n). This contradicts the uniqueness of the weak limit (consequence of Hahn–Banach theorem).

3 Outline the necessary changes in the proof of Theorem 4.15 in order to establish the existence of optimal contrtols to the boundary control problem (4.49)–(4.51) in [Tr].

Solution: The basic strategy is to follow the proof of Theorem 4.15:

1. Construct a minimizing sequence of controls $u_n \in L^\infty(\Gamma)$
2. Note that there is a corresponding sequence of unique bounded states $y_n \in H^1(\Omega) \cap C(\bar{\Omega})$ (see Theorems 4.6–4.8 in [Tr]). This sequence is bounded (Theorem 4.8)! Note that the linear part of the operator (that is, $-\Delta y + y$ with Neumann boundary conditions) is coercive, so there is no need to split the non-linearity as done in the book for the distributed control, where only $-\nabla y$ with Neumann boundary conditions is considered.
3. $H^1(\Omega)$ is a Hilbert space and therefore the sequence y_n contains a subsequence (can call it y_n again), converging weakly to $\bar{y} \in H^1(\Omega)$.
4. As in Theorem 4.15, we can use Rellich-Kondrachov compact embedding theorem (Theorem 7.4 in [Tr]) to go from the weak convergence in $H^1(\Omega)$ to a strong convergence in $L^2(\Omega)$.
5. Unlike in Theorem 4.15, this is not sufficient to conclude that $\bar{y} = y(\bar{u})$ because the non-linearity "lives" on the boundary now! This is really the largest deviation from the proof of Theorem 4.15 in [Tr].
   Let $T : H^1(\Omega) = W^{1,2}(\Omega) \to W^{1-1/2,2}(\Gamma) = H^{1/2}(\Gamma)$ be the trace operator (see for example Theorem 7.3 in [Tr]). Since T is a bounded linear operator, it follows that Ty_n converges weakly to $T\bar{y}$ in $H^{1/2}(\Gamma)$. Furthermore, $H^{1/2}(\Gamma)$ is compactly embedded into $L^2(\Gamma)$, see for example Theorem 7.1 in "The Hitchhiker's

guide to the fractional Sobolev spaces". Thus the non-linearity $b(\cdot, Ty_n(\cdot))$ will converge to the limit $b(\cdot, T\bar{y}(\cdot))$ owing to the continuity of Nemytskii operator, see Lemma 4.11 [Tr].

6. The rest of the proof follows more-or-less exactly the reasoning on p. 210 [Tr]. That is, $\bar{y} = y(\bar{u})$ and J is sequentially lower semi-continous with respect to the type of convergence that we need.