Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4183 Opt. II
Spring 2016

**Exercise set 8**

Please read sections 4.3–4.4 in [Tr].

1 Exercise 4.4 (ii) in [Tr]: Show that Nemytskii operator $y(\cdot) \mapsto \sin(y(\cdot))$ is Frechet differentiable from $L^{p_1}(0,T)$ into $L^{p_2}(0,T)$ whenever $1 \le p_2 < p_1 \le \infty$.

**Solution:** Let us put $\Psi(y) = \sin(y(\cdot))$. The directional derivative

$$\Psi'(y;h) = \lim_{\varepsilon \downarrow 0} \frac{\Psi(y+\varepsilon h) - \Psi(y)}{\varepsilon} = \cos(y(\cdot))h(\cdot),$$

is linear with respect $h$.

Thus we need to check that $\|\Psi(y+h) - \Psi(y) - \Psi'(y;h)\|_{L^{p_2}(0,T)}/\|h\|_{L^{p_1}(0,T)} \to 0$ when $\|h\|_{L^{p_1}(0,T)} \to 0$.

Let $\{h_n\}_{n=1}^{\infty} \in L^{p_1}(0,T)$ be a sequence converging to zero, and let us put $r_n = \Psi(y+h_n) - \Psi(y) - \Psi'(y;h_n)$. Additionally, let us chose an arbitrary small $\varepsilon > 0$. We will divide $\Omega = (0,T)$ into two parts: $\omega_n = \{\, x \in \Omega \mid |h_n(x)| < \varepsilon \,\}$ and $\Omega_n = \Omega \setminus \omega_n$. We will denote the characteristic functions of $\omega_n$ (respectively, $\Omega_n$) with $\chi_{\omega_n}$ (resp. $\chi_{\Omega_n}$.

On $\omega_n$ we can use the second order Taylor series expansion of $\sin(\cdot)$ combined with the fact that the second derivative of $\sin(\cdot)$ is bounded by 1 to get the estimate $|r_n(x)| \le |h_n(x)|^2/2 \le \varepsilon |h_n(x)|/2$.

On the other hand, $\Omega_n$ will be very small in measure for large $n$ (since convergence to zero in $L^{p_1}(0,T)$ implies convergence in measure). Furthermore, on $\Omega_n$ we can write $|r_n(x)| \le |\sin(y(x)+h_n(x)) - \sin(y(x))| + |\cos(y(x))h_n(x)| \le 2|h_n(x)|$, because $\sin(\cdot)$ is a Lipschitz function with constant 1 (derivative is bounded by 1).

Thus on $\omega_n$ we use the Taylor's series and Hölder's inequality to get the estimate:

$$\begin{aligned}
\|\chi_{\omega_n} r_n\|_{L^{p_2}(0,T)} &\le \varepsilon/2 \|\chi_{\omega_n} h_n\|_{L^{p_2}(0,T)} = \varepsilon/2 \biggl( \int_0^T \chi_{\omega_n}(x) |h_n(x)|^{p_2} \,\mathrm{d}x \biggr)^{1/p_2} \\
&\le \varepsilon/2 \bigl( \|\chi_{\omega_n}\|_{L^{p_1/(p_1-p_2)}(0,T)} \| |h_n|^{p_2} \|_{L^{p_1/p_2}(0,T)} \bigr)^{1/p_2} \\
&= \varepsilon/2 |\omega_n|^{(p_1-p_2)/(p_1 p_2)} \|h_n\|_{L^{p_1}(0,T)},
\end{aligned}$$

where $|\omega_n|$ denotes the Lebesgue measure of $\omega_n$, which is bounded by $|\Omega| = T$ in our case.

Similarly, on $\Omega_n$ we get

$$\|\chi_{\Omega_n} r_n\|_{L^{p_2}(0,T)} \le 2\|\chi_{\Omega_n} h_n\|_{L^{p_2}(0,T)} \quad \le 2|\Omega_n|^{(p_1-p_2)/(p_1 p_2)} \|h_n\|_{L^{p_1}(0,T)}.$$

In summary,

$$\begin{aligned}\|r_n\|_{L^{p_2}(0,T)} &= \|\chi_{\omega_n} r_n + \chi_{\Omega_n} r_n\|_{L^{p_2}(0,T)} \le \|\chi_{\omega_n} r_n\|_{L^{p_2}(0,T)} + \|\chi_{\Omega_n} r_n\|_{L^{p_2}(0,T)} \\ &\le (\varepsilon|\Omega|^{(p_1-p_2)/(p_1p_2)}/2 + 2|\Omega_n|^{(p_1-p_2)/(p_1p_2)})\|h_n\|_{L^{p_1}(0,T)},\end{aligned}$$

and therefore

$$0 \le \lim_{n\to\infty} \frac{\|r_n\|_{L^{p_2}(0,T)}}{\|h_n\|_{L^{p_1}(0,T)}} \le \varepsilon|\Omega|^{(p_1-p_2)/(p_1p_2)}/2,$$

since $\lim_{n\to\infty} |\Omega_n|^{(p_1-p_2)/(p_1p_2)} = 0$ for an arbitrary $\varepsilon > 0$. It remains to let $\varepsilon \to 0$ in the last inequality.

2 Compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ (Rellich–Kondrachov Theorem, Theorem 7.4 in [Tr]) plays an important role in the proof of Theorem 4.15 (existence of optimal controls for semi-linear elliptic PDEs). There are many other examples of compact embeddings.

Let $-\infty < a < b < +\infty$, and consider the spaces of continuous functions $C^0[a,b]$ and Hölder continuous functions $C^{0,\gamma}[a,b]$, $0 < \gamma \le 1$. These spaces are equipped with the norms

$$\begin{aligned}\|f\|_{C^0[a,b]} &= \sup_{x\in[0,T]} |f(x)|, \\ \|f\|_{C^{0,\gamma}[a,b]} &= \|f\|_{C^0[a,b]} + \sup_{x\neq y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.\end{aligned}$$

We will use Arzela–Ascoli characterization of relative compactness in $C^0[a,b]$ (it is not difficult to prove): The set $S \subset C^0[a,b]$ is relatively compact if and only if it is *bounded* and *equicontinuous*. That is, there is $M > 0$ such that $\forall f \in S : \|f\|_{C^0[a,b]} \le M$, and for every $\varepsilon > 0$ there is $\delta > 0$: $\forall f \in S, x, y \in [a,b] : |x-y| < \delta \implies |f(x)-f(y)| < \varepsilon$.

**a)** Show that every bounded subset in $C^{0,\gamma}[a,b]$ is bounded and equicontinuous in $C^0[a,b]$. Conclude that from any bounded sequence in $C^{0,\gamma}[a,b]$ one can extract a convergent sequence in $C^0[a,b]$.

**Solution:** Assume that $S \subset C^{0,\gamma}[a,b]$ is such that $\exists M > 0 : \forall f \in S, \|f\|_{C^{0,\gamma}[a,b]} \le M$. By definition $\|f\|_{C^0[a,b]} \le \|f\|_{C^{0,\gamma}[a,b]} \le M$ and thus $S$ is also a bounded set in $C^0[a,b]$. Furthermore from the definition of the norm we have that $|f(x)-f(y)| \le |x-y|^\gamma \|f\|_{C^{0,\gamma}[a,b]}$. Thus as long as $|x-y| < \delta$ it follows that $\forall f \in S : |f(x)-f(y)| < \delta^\gamma M$. Thus is is sufficient to choose $\delta = (\varepsilon/M)^{1/\gamma}$ in the definition of equicontinuity.

**b)** Show that any sequence $f_n \in C^{0,\gamma}[a,b]$, which converges weakly to some limit $\bar f \in C^{0,\gamma}[a,b]$, must satisfy $\|f_n - \bar f\|_{C^0[a,b]} \to 0$.
Hint: show the inclusion $(C^0[a,b])' \subset (C^{0,\gamma}[a,b])'$ for the dual spaces; use the fact that weakly convergent sequences are bounded (this is known as the uniform boundedness principle in functional analysis); then use the proof by contradiction and **a)**.

**Solution:** Suppose that $f_n \rightharpoonup \bar{f} \in C^{0,\gamma}[a,b]$. Weakly convergent sequences are bounded (uniform boundedness principle), and thus the set $\{f_n\}$ is relatively compact in $C^0[a,b]$ according to **a)** and Arzela-Ascolli theorem.
Let us now take $F \in (C^0[a,b])'$. Since $C^{0,\gamma}[a,b] \subset C^0[a,b]$ the function $F$ is defined and linear on $C^{0,\gamma}[a,b]$; furthermore $|F(f)| \leq \|F\|_{(C^0[a,b])'}\|f\|_{C^0[a,b]} \leq \|F\|_{(C^0[a,b])'}\|f\|_{C^{0,\gamma}[a,b]}$, $\forall f \in C^{0,\gamma}[a,b]$. As a result, $F \in (C^{0,\gamma}[a,b])'$ and $(C^0[a,b])' \subset (C^{0,\gamma}[a,b])'$. Since we know that $\forall F \in (C^{0,\gamma}[a,b])' : F(f_n) \to F(\bar{f})$, therefore this happens for all $F \in (C^0[a,b])'$ and $f_n \rightharpoonup \bar{f}$ in $C^0[a,b]$ as well.
Finally, assume that $\|f_n - \bar{f}\|_{C^0[a,b]} \not\to 0$, that is, for some $\varepsilon > 0$ there is a subsequence $n'$ of $n$ such that $\|f_{n'} - \bar{f}\|_{C^0[a,b]} \geq \varepsilon$. Since $\{f_{n'}\}$ is a subset of $\{f_n\}$, a relatively compact set in $C^0[a,b]$, we can extract a further subsequence $n''$ from it, such that $\|f_{n''} - \tilde{f}\|_{C^0[a,b]} \to 0$, for some $\tilde{f} \in C^0[a,b]$. Owing to the assumptions on $n'$, we have $\tilde{f} \neq \bar{f}$. Thus the subsequence $f_{n''}$ has two weak limits: $\tilde{f}$ (strong convergence implies weak) and $\bar{f}$ (as a subsequence of a weakly convergent sequence $f_n$). This contradicts the uniqueness of the weak limit (consequence of Hahn–Banach theorem).

3 Outline the necessary changes in the proof of Theorem 4.15 in order to establish the existence of optimal contrtols to the boundary control problem (4.49)–(4.51) in [Tr].

**Solution:** The basic strategy is to follow the proof of Theorem 4.15:

1. Construct a minimizing sequence of controls $u_n \in L^\infty(\Gamma)$
2. Note that there is a corresponding sequence of unique bounded states $y_n \in H^1(\Omega) \cap C(\bar{\Omega})$ (see Theorems 4.6–4.8 in [Tr]). This sequence is bounded (Theorem 4.8)! Note that the linear part of the operator (that is, $-\Delta y + y$ with Neumann boundary conditions) is coercive, so there is no need to split the non-linearity as done in the book for the distributed control, where only $-\nabla y$ with Neumann boundary conditions is considered.
3. $H^1(\Omega)$ is a Hilbert space and therefore the sequence $y_n$ contains a subsequence (can call it $y_n$ again), converging weakly to $\bar{y} \in H^1(\Omega)$.
4. As in Theorem 4.15, we can use Rellich-Kondrachov compact embedding theorem (Theorem 7.4 in [Tr]) to go from the weak convergence in $H^1(\Omega)$ to a strong convergence in $L^2(\Omega)$.
5. Unlike in Theorem 4.15, this is not sufficient to conclude that $\bar{y} = y(\bar{u})$ because the non-linearity "lives" on the boundary now! This is really the largest deviation from the proof of Theorem 4.15 in [Tr].
   Let $T : H^1(\Omega) = W^{1,2}(\Omega) \to W^{1-1/2,2}(\Gamma) = H^{1/2}(\Gamma)$ be the trace operator (see for example Theorem 7.3 in [Tr]). Since $T$ is a bounded linear operator, it follows that $Ty_n$ converges weakly to $T\bar{y}$ in $H^{1/2}(\Gamma)$. Furthermore, $H^{1/2}(\Gamma)$ is compactly embedded into $L^2(\Gamma)$, see for example Theorem 7.1 in "The Hitchhiker's guide to the fractional Sobolev spaces". Thus the non-linearity $b(\cdot, Ty_n(\cdot))$ will converge to the limit $b(\cdot, T\bar{y}(\cdot))$ owing to the continuity of Nemytskii operator, see Lemma 4.11 [Tr].
6. The rest of the proof follows more-or-less exactly the reasoning on p. 210 [Tr]. That is, $\bar{y} = y(\bar{u})$ and $J$ is sequentially lower semi-continous with respect to the type of convergence that we need.