



Please read sections 4.3–4.4 in [Tr].

- 1 Exercise 4.4 (ii) in [Tr]: Show that Nemytskii operator $y(\cdot) \mapsto \sin(y(\cdot))$ is Fréchet differentiable from $L^{p_1}(0, T)$ into $L^{p_2}(0, T)$ whenever $1 \leq p_2 < p_1 \leq \infty$.
- 2 Compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ (Rellich–Kondrachov Theorem, Theorem 7.4 in [Tr]) plays an important role in the proof of Theorem 4.15 (existence of optimal controls for semi-linear elliptic PDEs). There are many other examples of compact embeddings.

Let $-\infty < a < b < +\infty$, and consider the spaces of continuous functions $C^0[a, b]$ and Hölder continuous functions $C^{0,\gamma}[a, b]$, $0 < \gamma \leq 1$. These spaces are equipped with the norms

$$\|f\|_{C^0[a,b]} = \sup_{x \in [a,b]} |f(x)|,$$
$$\|f\|_{C^\gamma[a,b]} = \|f\|_{C^0[a,b]} + \sup_{x \neq y \in [a,b]} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.$$

We will use Arzela–Ascoli characterization of relative compactness in $C^0[a, b]$ (it is not difficult to prove): The set $S \subset C^0[a, b]$ is relatively compact if and only if it is *bounded* and *equicontinuous*. That is, there is $M > 0$ such that $\forall f \in S : \|f\|_{C^0[a,b]} \leq M$, and for every $\epsilon > 0$ there is $\delta > 0$: $\forall f \in S, x, y \in [a, b] : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

- a) Show that every bounded subset in $C^{0,\gamma}[a, b]$ is bounded and equicontinuous in $C^0[a, b]$. Conclude that from any bounded sequence in $C^{0,\gamma}[a, b]$ one can extract a convergent sequence in $C^0[a, b]$.
- b) Show that any sequence $f_n \in C^{0,\gamma}[a, b]$, which converges weakly to some limit $\bar{f} \in C^{0,\gamma}[a, b]$, must satisfy $\|f_n - \bar{f}\|_{C^0[a,b]} \rightarrow 0$.
Hint: show the inclusion $(C^0[a, b])' \subset (C^{0,\gamma}[a, b])'$ for the dual spaces; use the fact that weakly convergent sequences are bounded (this is known as the uniform boundedness principle in functional analysis); then use the proof by contradiction and **a**).

- 3 Outline the necessary changes in the proof of Theorem 4.15 in order to establish the existence of optimal controls to the boundary control problem (4.49)–(4.51) in [Tr].