TMA4183 Opt. II
Spring 2016

Exercise set 7

Norwegian University of Science and Technology
Department of Mathematical Sciences

Please read sections 4.1–4.2 in [Tr].

$\boxed{1}$ Let Assumptions 4.2 (i), (iii) and Assumptions 4.3 hold. Follow the ideas of Theorem 4.4 to show the existence and uniqueness of the solution to the semi-linear BPV for Laplace (or other uniformly elliptic operator as defined in (2.19)) operator with Dirichlet boundary conditions:

$$\begin{aligned} -\Delta y + d(x, y) &= f, &\quad \text{in } \Omega \\ y &= 0, &\quad \text{on } \Gamma. \end{aligned}$$

Solution: The proof literally follows the steps of Thm. 4.4, with the exception that in the present case the bilinear form $a(y, v) = \int_\Omega \nabla y(x) \cdot \nabla v(x) \, \mathrm{d}x$ on $[H_0^1(\Omega)]^2$ is used to define a coercive and strongly monotone operator $A_1 : H_0^1(\Omega) \to [H_0^1(\Omega)]^*$. The coercivity/strong monotonicity follows from the coercivity of a (see Section 2.3.1, and in particular Friedrichs inequality).

$\boxed{2}$ In several places in Section 4.2, for example in the proof of Theorem 4.5 or 4.7, the inequality

$$\|y\|_{L^\infty(\Gamma)} \le \|y\|_{L^\infty(\Omega)}, \qquad \forall y \in H^1(\Omega) \cap L^\infty(\Omega),$$

is used. Prove it! (this is Exercise 4.1 in [Tr], see the hints there).

Additional hints:

a) Show that if $\|y_n - y\|_{H^1(\Omega)} \to 0$ then $|y_n| \rightharpoonup |y|$, weakly in $H^1(\Omega)$. (It is sufficient to show that $|y_n| \rightharpoonup |y|$, weakly in $L^2(\Omega)$ and $D_i|y_n| \rightharpoonup D_i|y|$, weakly in $L^2(\Omega)$.)

b) The previous point shows that if $\|y_n - y\|_{H^1(\Omega)} \to 0$ then $P_{[-c,c]}(y_n) \rightharpoonup P_{[-c,c]}(y)$, weakly in $H^1(\Omega)$.

c) The trace operator trace : $H^1(\Omega) \to L^2(\Gamma)$ is *compact* and therefore maps weakly convergent sequences into strongly convergent ones. Use this information to prove the final result. Recall that convergence in L^2 implies convergence almost everyehere, up to a subsequence.

Solution:

(i): As suggested in the hint to the exercise let $c = \|y\|_{L^\infty(\Omega)}$, and further let $y_n \in C(\bar{\Omega}) \cap H^1(\Omega)$ be such that $\|y_n - y\|_{H^1(\Omega)} \to 0$. Let further $\hat{y}_n = P_{[-c,c]}(y_n)$.

(ii): $\hat{y}_n = \max\{-c, \min\{c, y_n\}\} \in C(\bar{\Omega})$ (recall: $\max\{y_1, y_2\} = (|y_1 - y_2| + y_1 + y_2)/2$). Thus $\|\mathrm{trace}(\hat{y}_n)\|_{L^\infty(\Gamma)} = \|\hat{y}_n|_\Gamma\|_{L^\infty(\Gamma)} \le c$, since $\sup_{x\in\bar{\Omega}} |\hat{y}_n(x)| \le c$ by construction.

(iii): Now we would like to show that the map $|\cdot| : H^1(\Omega) \to H^1(\Omega)$ (see exercises from the previous week) maps strongly convergent sequences into at least weakly convergent ones. Thus we want to show: $\forall z \in H^1(\Omega)$:

$$(z, |y_n| - |y|)_{H^1(\Omega)} = (z, |y_n| - |y|)_{L^2(\Omega)} + \sum_i (D_i z, D_i|y_n| - D_i|y|)_{L^2(\Omega)} \to 0.$$

The first summand goes to zero because $|\,|y_n| - |y|\,| \le |y_n - y|$. It remains to apply C-S inequality and use the strong convergence of y_n to y in $H^1(\Omega)$ (hence also in $L^2(\Omega)$).

For the second term, we use the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$. Let $C_0^\infty(\Omega) \ni \phi_k \to D_i z$, strongly in $L^2(\Omega)$. Then

$$\begin{aligned}
&|(D_i z, D_i|y_n| - D_i|y|)_{L^2(\Omega)}| \le |(D_i z - \phi_k, D_i|y_n| - D_i|y|)_{L^2(\Omega)}| + |(\phi_k, D_i|y_n| - D_i|y|)_{L^2(\Omega)}| \\
&\le \|D_i z - \phi_k\|_{L^2(\Omega)} \|\mathrm{sign}(y_n) D_i y_n - \mathrm{sign}(y) D_i(y)\|_{L^2(\Omega)} + |-(D_i\phi_k, |y_n| - |y|)_{L^2(\Omega)}| \\
&\le \|D_i z - \phi_k\|_{L^2(\Omega)} [\|\mathrm{sign}(y_n) D_i y_n\|_{L^2(\Omega)} + \|\mathrm{sign}(y) D_i y\|_{L^2(\Omega)}] + |(D_i\phi_k, |y_n| - |y|)_{L^2(\Omega)}| \\
&\le \|D_i z - \phi_k\|_{L^2(\Omega)} [\|D_i y_n\|_{L^2(\Omega)} + \|D_i y\|_{L^2(\Omega)}] + \|D_i\phi_k\|_{L^2(\Omega)} \||y_n| - |y|\|_{L^2(\Omega)}.
\end{aligned}$$

Since $D_i y_n \to D_i y$, strongly in $L^2(\Omega)$, the term $\|D_i y_n\|_{L^2(\Omega)} + \|D_i y\|_{L^2(\Omega)}$ in the square brackets is bounded. Let us first select and fix k so large, that the term $\|D_i z - \phi_k\|_{L^2(\Omega)} [\|D_i y_n\|_{L^2(\Omega)} + \|D_i y\|_{L^2(\Omega)}]$ is small for all n. Then, since $|y_n| \to |y|$, strongly in $L^2(\Omega)$, it only remains to choose n so large that the last term is small.

(iv): We now use the fact that trace maps weakly convergent sequences in $H^1(\Omega)$ into strongly convergent sequences in $L^2(\Gamma)$. Thus $y_n \to y$, strongly in $H^1(\Omega)$ $\implies$ $P_{[-c,c]}(y_n) \rightharpoonup P_{[-c,c]}(y)$, weakly in $H^1(\Omega)$ $\implies$ $\|\mathrm{trace}(P_{[-c,c]}(y_n)) - \mathrm{trace}(P_{[-c,c]}(y))\|_{L^2(\Gamma)} = \|\mathrm{trace}(\hat{y}_n) - \mathrm{trace}(y)\|_{L^2(\Gamma)} \to 0$. Thus $\hat{y}_{n_k}|_\Gamma \to \mathrm{trace}(y)$ for some subsequence y_{n_k}, almost everywhere on Γ, and therefore $\mathrm{trace}(y) \le c$, a.e. on Γ.