Exercise set 5^{num}

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In this exercise we compare three numerical algorithms: conditional gradient, projected gradient, and primal-dual active set method for solving the a linear-quadratic control problem for Laplace equation with bound control constraints.

Please read section 2.9 in [Tr] on how to construct test examples.

We will try to solve the problem

$$\begin{aligned}&\text{minimize } J(y,u) = \frac{1}{2}\int_\Omega (y(x) - y_\Omega(x))^2\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega u^2(x)\,\mathrm{d}x,\\ &\text{subject to } u \in U_{\mathrm{ad}},\end{aligned}$$

where $U_{\mathrm{ad}} = \{\, u \in L^2(\Omega) \mid u_a(x) \le u(x) \le u_b(x) \,\}$ and $y \in H_0^1(\Omega)$ solves the Laplace equation

$$\begin{aligned}-\Delta y &= \beta(x)u(x) + f_0,\\ y|_\Gamma &= 0,\end{aligned}$$

where $f_0 \in L^2(\Omega)$ is a given function. Note that in this case the solution operator $u \mapsto y$ can be written as $y_0 + Su$, where $y_0 \in H_0^1(\Omega)$ is a fixed function solving the non-homogeneous problem

$$\begin{aligned}-\Delta y_0 &= f_0,\\ y_0|_\Gamma &= 0.\end{aligned}$$

This is a purely theoretical consideration; you do not need to construct y_0 in practice. Basically all the previously developed theory holds; e.g. the Frechet derivative (with respect to u) $(y_0 + Su)' = S$.

I suggest starting with some simple Ω; in this document I will assume that $\Omega = [0,1]^2$.

Test example

We have to determine the problem data y_Ω, u_a, u_b etc, for which we know an analytic solution. In this way we can actually verify that (or whether) our algorithm converges to this solution when we refine the discretization.

Let us for example put $\bar{u}$ to be the characteristic function of a ball of radius $1/4$ in the middle of the unit square. We can now take some $\bar{y} \in H_0^1(\Omega)$, for example, $\bar{y}(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2)$, and $\beta \in L^\infty(\Omega)$, for example $\beta = 1 - 2\bar{u}$ from which we compute $f_0 = -\Delta\bar{y} - \beta\bar{u}$.

Now we need to determine $\bar{p} \in H_0^1(\Omega)$, λ, u_a, u_b such that

$$\int_\Omega (\beta\bar{p} + \lambda\bar{u})(u - \bar{u})\,\mathrm{d}x \geq 0, \qquad \forall u \in U_{\mathrm{ad}}.$$

Let us take $u_a \equiv 0$, $u_b \equiv 1$, then it is sufficient to choose $\bar{p}$ such that $\beta\bar{p} + \lambda\bar{u} = \beta\bar{p} \geq 0$ when $\bar{u} = u_a$ and $\beta\bar{p} + \lambda\bar{u} \leq 0$ when $\bar{u} = u_b$. Note that because of our choice of β we can select $\bar{p} \geq 0$, $\bar{p} > 0$ when $\bar{u} = u_b$ and then select $\lambda > 0$ such that $\bar{p} > \lambda$ when $\bar{u} = u_b$. From this perspective, for example $\bar{p}(x_1, x_2) = x_1(1 - x_1)x_2(1 - x_2)$ and $\lambda = 1/4 * 3/4 * 1/4 * 3/4$ should do the trick.

Finally, we can put $y_\Omega = \bar{y} + \Delta\bar{p}$.

$\boxed{1}$ Fill out the details for the construction of the test example. Make sure that $(\bar{u}, \bar{y}, \bar{p})$ satisfy the (necessary and sufficient) optimality conditions.

Numerical algorithms

In your implementaiton you could either use Matlab/cell centered polygonal discretization briefly described in Exercise 4^{num} or FEniCS/Python implementation.

$\boxed{2}$

a) Implement the conditional gradient algorithm with exact linesearch. Verify your implementation by making sure that the code converges to the right solution on the test example.

Study how the number of iterations varies with changes in the required accuracy and the "fineness" of the discretization.

b) Implement the projected gradient algorithm with backtracking linesearch.

Verify it and study its behaviour as with the conditional gradient algorithm.

c) Implement the primal-dual active set algorithm.

Verify it and study its behaviour as with the conditional gradient algorithm.