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Exercise set 4

Reading material: Section 2.8–2.10 from [Tröltsch].

1 Exercise 2.11 [Tr]:
Let real numbers u_a, u_b, β, p and some $\lambda > 0$ be given. Solve the quadratic optimization problem in $\mathbb{R}$,

$$\min_{v\in[u_a,u_b]} \left\{ \beta p v + \frac{\lambda}{2} v^2 \right\},$$

by deriving a projection formula of the type (2.58) on page 70.

Solution: Let $f(v) = \beta p v + \lambda v^2/2$. Since f is convex and differentiable, $\bar{u} \in [u_a, u_b]$ is an optimal solution iff $f'(\bar{u})(u - \bar{u}) = [\beta p + \lambda \bar{u}](u - \bar{u}) \geq 0$, $\forall u \in [u_a, u_b]$.

If $\beta p + \lambda \bar{u} > 0$ the variational inequality can only hold when $u - \bar{u} \geq 0$ $\forall u \in [u_a, u_b]$, that is, when $\bar{u} = u_a$. In this case we have $\lambda u_a = \lambda \bar{u} > -\beta p$ or $u_a > -\beta p/\lambda$. Thus $\bar{u} = u_a = \max\{u_a, -\beta p/\lambda\} = \min\{u_b, \max\{u_a, -\beta p/\lambda\}\} = P_{[u_a,u_b]}(-\beta p/\lambda)$.

Similarly if $\beta p + \lambda \bar{u} < 0$ the variational inequality can only hold when $\bar{u} = u_b$. In this case we have $\lambda u_b = \lambda \bar{u} < -\beta p$ or $u_b < -\beta p/\lambda$. Thus $\bar{u} = u_b = \min\{u_b, -\beta p/\lambda\} = \min\{u_b, \max\{u_a, -\beta p/\lambda\}\} = P_{[u_a,u_b]}(-\beta p/\lambda)$.

Finally, if $\beta p + \lambda \bar{u} = 0$ the variational inequality holds, but we have the restriction that $[u_a, u_b] \ni \bar{u} = -\beta p/\lambda$. Thus $\bar{u} = -\beta p/\lambda = \min\{u_b, \max\{u_a, -\beta p/\lambda\}\} = P_{[u_a,u_b]}(-\beta p/\lambda)$.

2 Exercise 2.19 [Tr]:
Apply the formal Lagrange method to the problem with boundary control of Dirichlet type:

$$\min \int_\Omega |y - y_\Omega|^2 \,\mathrm{d}x + \lambda \int_\Gamma |u|^2 \,\mathrm{d}s =: J(y,u),$$

subject to

$$-\Delta y = 0,$$
$$y|_\Gamma = u,$$
$$-1 \leq u(x) \leq 1.$$

Solution:

We define the Lagrangian:

$$\begin{aligned} L(y,u,p_1,p_2) &= J(y,u) + \int_\Omega (\Delta y) p_1 \,\mathrm{d}x - \int_\Gamma (y-u)p_2 \,\mathrm{d}s \\ &= J(y,u) - \int_\Omega \nabla y \cdot \nabla p_1 \,\mathrm{d}x + \int_\Gamma p_1 \nu \cdot \nabla y \,\mathrm{d}s - \int_\Gamma (y-u)p_2 \,\mathrm{d}s \\ &= J(y,u) + \int_\Omega y \cdot \Delta p_1 \,\mathrm{d}x - \int_\Gamma y\nu \cdot \nabla p_1 \,\mathrm{d}s + \int_\Gamma p_1 \nu \cdot \nabla y \,\mathrm{d}s - \int_\Gamma (y-u)p_2 \,\mathrm{d}s \end{aligned}$$

Then one proceeds in the same way as outlined in Section 2.10 of the book. By taking the variation of L with respect to y in any direction $h \in C_0^\infty(\Omega)$ we get

$$0 = L_y' h = 2\int_\Omega (y - y_\Omega)h + \int_\Omega \Delta p_1 h,$$

and therefore $-\Delta p_1 = 2(y - y_\Omega)$, in Ω. Next one requires that $h|_\Gamma = 0$ but $\nu \cdot \nabla h$ is allowed to vary. This "picks up" the term

$$0 = L_y' h = \int_\Gamma p_1 \nu \cdot \nabla h \,\mathrm{d}s,$$

and therefore $p_1|_\Gamma = 0$. Finally, we take $h \in W^{1,2}(\Omega)$ to be an arbitrary direction, which then gives us

$$0 = L_y' h = -\int_\Gamma h\nu \cdot \nabla p_1 \,\mathrm{d}s - \int_\Gamma h p_2 \,\mathrm{d}s,$$

and therefore $p_2 = -\nu \cdot \nabla p_1$ on Γ.

Finally, taking the variation w.r.t. to u, we get that at any optimal solution $-1 \le \bar{u} \le 1$ the following variational inequality must hold:

$$L_u'(u - \bar{u}) = \int_\Gamma (2\lambda \bar{u} + p_2)(u - \bar{u}) \ge 0, \qquad \forall -1 \le u(x) \le 1.$$

3 Exercise 2.15 [Tr]:
Derive the necessary optimality conditions for the linear optimal control problem on page 79:

$$\min\ J(y,u,v) := \int_\Omega (a_\Omega y + \lambda_\Omega v)\,\mathrm{d}x + \int_\Gamma (a_\Gamma y + \lambda_\Gamma u)\,\mathrm{d}s,$$

subject to

$$\begin{aligned} -\Delta y &= \beta_\Omega v, && \text{in } \Omega, \\ \partial_\nu y + \alpha y &= \beta_\Gamma u, && \text{on } \Gamma, \\ v_a(x) \le v(x) &\le v_b(x), && \text{a.e. in } \Omega, \\ u_a(x) \le u(x) &\le u_b(x), && \text{a.e. on } \Gamma, \end{aligned}$$

Solution:

Let $(\bar{y}, \bar{u}, \bar{v}) \in H^1(\Omega) \times U_{ad} \times V_{ad}$ be an optimal solution (state, distributed control, boundary control). Then for all $(y, u, v) \in H^1(\Omega) \times U_{ad} \times V_{ad}$ we necessarily have:

$$
\begin{aligned}
0 &\leq J(y,u,v) - J(\bar{y},\bar{u},\bar{v}) = J(y-\bar{y}, u-\bar{u}, v-\bar{v}) \\
&= \int_\Omega (a_\Omega(y-\bar{y}) + \lambda_\Omega(v-\bar{v}))\,\mathrm{d}x + \int_\Gamma (a_\Gamma(y-\bar{y}) + \lambda_\Gamma(u-\bar{u})),
\end{aligned}
$$

$$
\begin{aligned}
-\Delta(y-\bar{y}) &= \beta_\Omega(v-\bar{v}), \\
\partial_\nu(y-\bar{y}) + \alpha(y-\bar{y}) &= \beta_\Gamma(u-\bar{u}),
\end{aligned}
$$

Note that $y - \bar{y} \in H^1(\Omega)$ solves the variational problem

$$
\begin{aligned}
\int_\Omega \nabla(y-\bar{y}) \cdot \nabla z - \int_\Gamma \partial_\nu(y-\bar{y}) z &= \int_\Omega \beta_\Omega(v-\bar{v}) z, \qquad \forall z \in H^1(\Omega), \text{or} \\
\int_\Omega \nabla(y-\bar{y}) \cdot \nabla z + \int_\Gamma \alpha(y-\bar{y}) z &= \int_\Omega \beta_\Omega(v-\bar{v}) z + \int_\Gamma \beta_\Gamma(u-\bar{u}) z \qquad \forall z \in H^1(\Omega).
\end{aligned}
$$

If we now introduce the adjoint problem (see p. 80 - can be derived using the Formal Lagrange Method):

$$
\begin{aligned}
-\Delta p &= a_\Omega, &&\text{in } \Omega, \\
\partial_\nu p + \alpha p &= a_\Gamma, &&\text{on } \Gamma,
\end{aligned}
$$

with the variational formulation

$$
\int_\Omega \nabla p \cdot \nabla q + \int_\Gamma \alpha p q = \int_\Omega a_\Omega q + \int_\Gamma a_\Gamma q \qquad \forall q \in H^1(\Omega).
$$

If we put $z = p$, $q = y - \bar{y}$ in the above then

$$
\int_\Omega a_\Omega(y-\bar{y}) + \int_\Gamma a_\Gamma(y-\bar{y}) = \int_\Omega \beta_\Omega(v-\bar{v}) p + \int_\Gamma \beta_\Gamma(u-\bar{u}) p.
$$

Thus the variational inequality $J(y-\bar{y}, u-\bar{u}, v-\bar{v}) \geq 0$ becomes

$$
\int_\Omega (\beta_\Omega p + \lambda_\Omega)(v-\bar{v}) + \int_\Gamma (\beta_\Gamma p + \lambda_\Gamma)(u-\bar{u}) \geq 0, \quad \forall (v,u) \in V_{ad} \times U_{ad}.
$$

Note that the problem is convex and therefore these conditions are also sufficient.