



Reading material: Sections 2.5–2.7 from [Tröltzsch].

1 Exercise 2.11 [Tr]:

- a) Show that $f(u) = \sin(u(1))$ is continuously Frechet differentiable in $C[0, 1]$ (i.e., it is Frechet differentiable and the derivative is a continuous function).

Solution: We compute the first variation:

$$\delta f(u, h) = \lim_{\varepsilon \downarrow 0} \frac{\sin(u(1) + \varepsilon h(1)) - \sin(u(1))}{\varepsilon} = \frac{d}{d\varepsilon} [\sin(u(1) + \varepsilon h(1))] |_{\varepsilon=0} = \cos(u(1))h(1).$$

Since $\delta f(u, h)$ is linear with respect to h and also is bounded:

$$|\delta f(u, h)| \leq |\cos(u(1))| |h(1)| \leq |\cos(u(1))| \sup_{x \in [0, 1]} |h(x)| = |\cos(u(1))| \|h\|_{C[0, 1]},$$

f is in fact Gateaux differentiable. Furthermore

$$\begin{aligned} & \lim_{\|h\|_{C[0, 1]} \rightarrow 0} \frac{|\sin(u(1) + h(1)) - \sin(u(1)) - \cos(u(1))h(1)|}{\|h\|_{C[0, 1]}} \\ & \leq \lim_{\|h\|_{C[0, 1]} \rightarrow 0} \frac{|-\sin(u(1) + \theta h(1))h^2(1)/2|}{\|h\|_{C[0, 1]}} \leq \lim_{\|h\|_{C[0, 1]} \rightarrow 0} \frac{\|h\|_{C[0, 1]}^2}{2\|h\|_{C[0, 1]}} = 0, \end{aligned}$$

$\theta \in [0, 1]$, where we used a second order Taylor series expansion of \sin and the fact that $|\sin| \leq 1$. Thus f is Frechet differentiable.

Finally we have that

$$\begin{aligned} \|f'(u_1) - f'(u_2)\|_{\mathcal{L}(C[0, 1], \mathbb{R})} & \leq \sup_{h \neq 0} \frac{|[\cos(u_1(1)) - \cos(u_2(1))]h(1)|}{\|h\|_{C[0, 1]}} \\ & \leq |\sin(u_2(1) + \theta u_1(1))(u_1(1) - u_2(1))| \leq \|u_1 - u_2\|_{C[0, 1]}. \end{aligned}$$

Thus f' is continuous (in fact, Lipschitz continuous).

- b) Show that $f(u) = \|u\|_H^2$ is continuously Frechet differentiable in H , where H is an arbitrary Hilbert space.

Solution:

The representation $f(u + h) - f(u) = 2(u, h) + \|h\|_H^2$ immediately implies that $f'(u)h = 2(u, h)$ is a Frechet derivative. Its continuity is ensured by the continuity of the inner product (Cauchy–Schwarz).

- c) $C[0,1]$ is everywhere dense in $L^2(0,1)$. Does this imply that $f(u)$ in a) is continuously Frechet differentiable on $L^2(0,1)$?

Solution:

No, Frechet differentiable functions must be continuous, whereas $u(1)$ is not continuous on $L^2[0,1]$. (E.g. consider the sequence $u_k(x) = x^k \rightarrow 0$ in L^2 .) I.e. $C[0,1]$ is dense in $L^2[0,1]$ only with respect to L^2 -norm, not C -norm.

- 2] Let H be a Hilbert space, $a : H \times H \rightarrow \mathbb{R}$ be a symmetric, bounded and coercive bilinear form, and $L \in H'$ be a bounded linear functional. Let us define $f : H \rightarrow \mathbb{R}$ by $f(u) = a(u,u)/2 - L(u)$. Show that f is continuously Frechet differentiable on H . Express the condition $f' = 0$ as a variational problem.

Solution:

Similarly to the inner product we have the expansion $f(u+h) - f(u) = a(u,h) + a(h,h)/2 - L(h)$ (we have used the symmetry here). Owing to the boundedness of a we immediately get that $f'(u)h = a(u,h) - L(h)$ and that f' is a continuous function. As a result, $f'(u) = 0$ in $H' \iff f'(u)h = 0, \forall h \in H \iff a(u,h) = L(h), \forall h \in H$.

- 3] Exercise 2.10 [Tr]:

Suppose that Y and U are Hilbert spaces, and let $y_d \in U$, $\lambda \geq 0$, and operator $S \in \mathcal{L}(U, Y)$ be given. Show that the functional

$$f(u) = \|Su - y_d\|_Y^2 + \lambda \|u\|_U^2$$

is strictly convex if $\lambda > 0$ or S is injective.

Solution:

Take arbitrary $u_1 \neq u_2$, and put $u = (u_1 + u_2)/2$. We will show that $f(u) < (f(u_1) + f(u_2))/2$. Then, since any point on the line segment $[u_1, u_2]$ can be written as a convex combination of either u_1 and u or u and u_2 , this will imply strict convexity, e.g., if $0 < \lambda < 1/2$ then

$$\begin{aligned} f(\lambda u_1 + (1-\lambda)u_2) &= f(2\lambda u + (1-2\lambda)u_2) \leq 2\lambda f(u) + (1-2\lambda)f(u_2) \\ &< \lambda(f(u_1) + f(u_2)) + (1-2\lambda)f(u_2) = \lambda f(u_1) + (1-\lambda)f(u_2), \end{aligned}$$

and similarly for $1/2 < \lambda < 1$.

Let us now compute these quantities in our case:

$$\begin{aligned} f(u) &= \|1/2(Su_1 - y_d) + 1/2(Su_2 - y_d)\|_Y^2 + \lambda \|1/2u_1 + 1/2u_2\|_U^2 \\ &= 1/4[\|Su_1 - y_d\|_Y^2 + \|Su_2 - y_d\|_Y^2 + \lambda \|u_1\|_U^2 + \lambda \|u_2\|_U^2] \\ &\quad + 1/2(Su_1 - y_d, Su_2 - y_d)_Y + \lambda/2(u_1, u_2)_U, \\ (f(u_1) + f(u_2))/2 &= 1/2[\|Su_1 - y_d\|_Y^2 + \|Su_2 - y_d\|_Y^2 + \lambda \|u_1\|_U^2 + \lambda \|u_2\|_U^2]. \end{aligned}$$

Therefore

$$(f(u_1) + f(u_2))/2 - f(u) = 1/4\|S(u_1 - u_2)\|_Y^2 + \lambda/4\|u_1 - u_2\|_U^2 \geq 0.$$

(This is the same "trick" as we used in exercise 1a) in last week.) Note that the equality cannot be attained for $u_1 \neq u_2$ if either $\lambda > 0$ or S is injective.

- 4 Let Ω be a bounded Lipschitz domain and consider the “identity” operator $i : W^{1,2}(\Omega) \rightarrow L^2(\Omega)$, defined as $i(u) = u$. Describe its Hilbert space adjoint i^* , that is, for a given $v \in L^2(\Omega)$ state the variational problem solved by $i^*(v)$. Find the PDE/boundary value problem, whose weak solution is given by $i^*(v)$.

Solution:

Then for the Hilbert space adjoint we have:

$$(i^*(v), u)_{W^{1,2}(\Omega)} = (v, i(u))_{L^2(\Omega)}, \quad \forall u \in W^{1,2}(\Omega), v \in L^2(\Omega).$$

Thus we can define a symmetric, continuous, and coercive bilinear form $a(u_1, u_2) = (u_1, u_2)_{W^{1,2}(\Omega)}$ on $[W^{1,2}(\Omega)]^2$ and a linear bounded functional $L_v(u) = (v, u)_{L^2(\Omega)}$ on $W^{1,2}(\Omega)$. Then $i^*(v)$ is the unique solution of the variational problem $a(i^*(v), u) = L_v(u), \forall u \in W^{1,2}(\Omega)$.

- 5 Let H be a Hilbert space.

a) Exercise 2.8 [Tr]:

Assume that $H \ni u_n \rightharpoonup u \in H$ and $H \ni v_n \rightarrow v \in H$. Show that $(u_n, v_n) \rightarrow (u, v)$.

Solution:

$$|(u_n, v_n) - (u, v)| \leq |(u_n - u, v_n)| + |(u, v_n - v)| \leq [\sup_k \|v_k\|_H] \|u_n - u\|_H |(u, v_n - v)|.$$

Since v_k converges weakly, it is also bounded (this follows from the uniform boundedness principle (Banach-Steinhaus theorem) mentioned on p. 44 in [Tr].) Thus the first summand must converge to zero because u_n converges strongly to u in H , and the last term goes to zero because v_n converges weakly to u .

b) Construct an example where $H \ni u_n \rightharpoonup u \in H$ and $H \ni v_n \rightharpoonup v \in H$, but $(u_n, v_n) \not\rightarrow (u, v)$. (Hint: it is sufficient to consider $u_n = v_n$.)

Solution:

Take $u_n = v_n = e_n$ for an orthonormal basis in any Hilbert space (see example (iii) in [Tr], p. 44). Then $u_n \rightharpoonup 0$ but $\|u_n\|_H = 1$.

c) Show that if $H \ni u_n \rightharpoonup u \in H$ and in addition $\|u_n\| \rightarrow \|u\|$ then also $u_n \rightarrow u$.

Solution:

$$\|u_n - u\|_H^2 = \|u_n\|^2 + \|u\|^2 - 2(u, u_n) \rightarrow 2\|u\|^2 - 2(u, u) = 0.$$

- 6 Establish the following generalization of Lax–Milgram theorem. Let U, V be two Hilbert spaces and consider a bilinear form a on $U \times V$, satisfying the following conditions:

1. Continuity: there is a constant $\alpha > 0$ such that $a(u, v) \leq \alpha \|u\|_U \|v\|_V$.

2. Inf-sup, or Ladyzhenskaya–Babuška–Brezzi (LBB) condition: there is a constant $\beta > 0$ such that

$$\inf_{u \in U \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{a(u, v)}{\|u\|_U \|v\|_V} \geq \beta.$$

3. For all $v \in V \setminus \{0\}$ it holds that $\sup_{u \in U} |a(u, v)| > 0$.

Then for every bounded linear functional $f \in V'$ the variational problem

$$a(u, v) = f(v), \quad \forall v \in V,$$

admits a unique solution $\hat{u} \in U$. This solution satisfies the estimate $\|\hat{u}\|_U \leq \beta^{-1} \|f\|_{V'}$.

Sketch of the proof:

1. Show that the functional $F_u : V \rightarrow \mathbb{R}$ defined by $F_u(v) = a(u, v)$ is bounded and linear.

Solution: Linearity follows from that of a ; $|F_u(v)| \leq |a(u, v)| \leq \alpha \|u\|_U \|v\|_V \implies \|F_u\|_{V'} \leq \alpha \|u\|_U$.

2. Define an operator $A : U \rightarrow V$ by $Au = R_V^{-1}(F_u)$, where $R_V : V \rightarrow V'$ is the Riesz' map $V \ni v \mapsto (v, \cdot)_V \in V'$. Show that this operator is linear and bounded.

Solution: F_u is linear w.r.t. u because $a(u, v)$ is; furthermore $\|Au\|_V = \|R_V^{-1}(F_u)\|_V = \|F_u\|_{V'} \leq \alpha \|u\|_U$, where we have used the fact that R_V is an isometry. Thus $\|A\|_{\mathcal{L}(U, V)} \leq \alpha$.

3. Use LBB condition to show that A is bounded from below, that is $\|Au\|_V \geq \beta \|u\|_U$.

Solution:

$$\|Au\|_V = \|R_V^{-1}(F_u)\|_V = \|F_u\|_{V'} = \sup_{v \neq 0} |F_u(v)| / \|v\|_V = \sup_{v \neq 0} |a(u, v)| / \|v\|_V \geq \beta \|u\|_U$$

owing to the LBB-condition.

4. From the previous condition show that A admits a bounded inverse A^{-1} acting from the range $R(A) \subset V$ of A to U .

Solution: A is linear and injective; indeed $\|Au_1 - Au_2\|_V \geq \beta \|u_1 - u_2\|_U$ and therefore is invertible on its range $R(A)$.

5. Again using the boundedness of A from below, show that $R(A)$ is closed in V .

Solution: Let $v_k = Au_k$ be a Cauchy sequence in $R(A)$. Then $\|v_i - v_j\|_V = \|A(u_i - u_j)\|_V \geq \beta \|u_i - u_j\|_U$. Thus u_k is Cauchy in U , and since U is a Hilbert space, it has a limit \bar{u} . Owing to the continuity of A : $\|v_k - A\bar{u}\|_V \leq \|A\| \|u_k - \bar{u}\|_U \rightarrow 0$. Thus $v_k \rightarrow A\bar{u}$ and $R(A)$ is closed.

6. Use the last assumption of the theorem to show that $R(A)^\perp = \{0\}$; therefore $R(A) = V$.

Solution: Let $v \in R(A)^\perp$. Then $(Au, v)_V = 0, \forall u \in U$. But $(Au, v) = R_V(Au)[v] = F_u(v) = a(u, v)$. Owing to the assumption 3 this implies that $v = 0$, and thus $R(A) = V$.

7. Conclude the proof of existence by showing that $\hat{u} = A^{-1}R_V^{-1}f$ solves the variational problem.

Solution:

Thus we have $R(A) = V$ and therefore the inverse $A^{-1} : V \rightarrow U$. Let us put $\hat{u} = A^{-1}R_V^{-1}f$; then $A\hat{u} = R_V^{-1}f$, $R_V(A\hat{u}) = f$, $R_V(A\hat{u})[v] = (Au, v)_V = a(u, v) = f(v)$, for all $v \in V$.

8. The norm bound on the solution follows from the boundedness of A^{-1} . The uniqueness of the solution follows from this bound and the linearity.

Solution:

Note that the lower bound $\|Au\|_V \geq \beta\|u\|_U$ implies $\|A^{-1}v\|_U \leq \beta^{-1}\|v\|_V$ by putting $u = A^{-1}v$. We thus have $\|\hat{u}\|_U \leq \|A^{-1}\| \|R_V^{-1}f\|_V \leq \beta^{-1}\|f\|_{V'}$.

If we had two solutions u_1, u_2 , then their difference would solve the homogeneous problem, and therefore the previous bound would imply $\|u_1 - u_2\|_U \leq \beta^{-1}\|0\|_{V'} = 0$.