



Reading material: Sections 2.5–2.7 from [Tröltzsch].

1 Exercise 2.11 [Tr]:

- a) Show that $f(u) = \sin(u(1))$ is continuously Frechet differentiable in $C[0, 1]$ (i.e., it is Frechet differentiable and the derivative is a continuous function).
- b) Show that $f(u) = \|u\|_H^2$ is continuously Frechet differentiable in H , where H is an arbitrary Hilbert space.
- c) $C[0, 1]$ is everywhere dense in $L^2(0, 1)$. Does this imply that $f(u)$ in a) is continuously Frechet differentiable on $L^2(0, 1)$?

2 Let H be a Hilbert space, $a : H \times H \rightarrow \mathbb{R}$ be a symmetric, bounded and coercive bilinear form, and $L \in H'$ be a bounded linear functional. Let us define $f : H \rightarrow \mathbb{R}$ by $f(u) = a(u, u)/2 - L(u)$. Show that f is continuously Frechet differentiable on H . Express the condition $f' = 0$ as a variational problem.

3 Exercise 2.10 [Tr]:

Suppose that Y and U are Hilbert spaces, and let $y_d \in U$, $\lambda \geq 0$, and operator $S \in \mathcal{L}(U, Y)$ be given. Show that the functional

$$f(u) = \|Su - y_d\|_Y^2 + \lambda \|u\|_U^2$$

is strictly convex if $\lambda > 0$ or S is injective.

4 Let Ω be a bounded Lipschitz domain and consider the “identity” operator $i : W^{1,2}(\Omega) \rightarrow L^2(\Omega)$, defined as $i(u) = u$. Describe its Hilbert space adjoint i^* , that is, for a given $v \in L^2(\Omega)$ state the variational problem solved by $i^*(v)$. Find the PDE/boundary value problem, whose weak solution is given by $i^*(v)$.

5 Let H be a Hilbert space.

a) Exercise 2.8 [Tr]:

Assume that $H \ni u_n \rightarrow u \in H$ and $H \ni v_n \rightarrow v \in H$. Show that $(u_n, v_n) \rightarrow (u, v)$.

- b) Construct an example where $H \ni u_n \rightharpoonup u \in H$ and $H \ni v_n \rightharpoonup v \in H$, but $(u_n, v_n) \not\rightarrow (u, v)$. (Hint: it is sufficient to consider $u_n = v_n$.)
- c) Show that if $H \ni u_n \rightharpoonup u \in H$ and in addition $\|u_n\| \rightarrow \|u\|$ then also $u_n \rightarrow u$.

6 Establish the following generalization of Lax–Milgram theorem. Let U, V be two Hilbert spaces and consider a bilinear form a on $U \times V$, satisfying the following conditions:

1. Continuity: there is a constant $\alpha > 0$ such that $a(u, v) \leq \alpha \|u\|_U \|v\|_V$.
2. Inf-sup, or Ladyzhenskaya–Babuška–Brezzi (LBB) condition: there is a constant $\beta > 0$ such that

$$\inf_{u \in U \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{a(u, v)}{\|u\|_U \|v\|_V} \geq \beta.$$

3. For all $v \in V \setminus \{0\}$ it holds that $\sup_{u \in U} |a(u, v)| > 0$.

Then for every bounded linear functional $f \in V'$ the variational problem

$$a(u, v) = f(v), \quad \forall v \in V,$$

admits a unique solution $\hat{u} \in U$. This solution satisfies the estimate $\|\hat{u}\|_U \leq \beta^{-1} \|f\|_{V'}$.

Sketch of the proof:

1. Show that the functional $F_u : V \rightarrow \mathbb{R}$ defined by $F_u(v) = a(u, v)$ is bounded and linear.
2. Define an operator $A : U \rightarrow V$ by $Au = R_V^{-1}(F_u)$, where $R_V : V \rightarrow V'$ is the Riesz' map $V \ni v \mapsto (v, \cdot)_V \in V'$. Show that this operator is linear and bounded.
3. Use LBB condition to show that A is bounded from below, that is $\|Au\|_V \geq \beta \|u\|_U$.
4. From the previous condition show that A admits a bounded inverse A^{-1} acting from the range $R(A) \subset V$ of A to U .
5. Again using the boundedness of A from below, show that $R(A)$ is closed in V .
6. Use the last assumption of the theorem to show that $R(A)^\perp = \{0\}$; therefore $R(A) = V$.
7. Conclude the proof of existence by showing that $\hat{u} = A^{-1}R_V^{-1}f$ solves the variational problem.
8. The norm bound on the solution follows from the boundedness of A^{-1} . The uniqueness of the solution follows from this bound and the linearity.