Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4183 Opt. II
Spring 2016

Exercise set 2

Reading material: Sections 2.1–2.4 from [Tröltsch].

1 On monday we have proved the following result: let C be a non-empty closed and convex subset of a Hilbert space H. Then there is $\bar{x} \in C$ with the smallest length, that is, $\|\bar{x}\| = \min_{x\in C} \|x\|$. Now we will use this result to prove Riesz representation theorem: for any $f \in H'$ there is a unique $x_f \in H$ such that $\forall x \in H : f(x) = (x, x_f)$ and $\|f\|_{H'} = \|x_f\|_H$.

Since this result is trivial for $f \equiv 0$, we will assume that $\|f\| \neq 0$.

a) Let C be a non-empty closed convex subset of H. Show that there is only one vector of shortest length in it. Hint: $2(x, y) \leq \|x\|^2 + \|y\|^2$ with equality iff $\|x - y\|^2 = 0$.

Solution:
Suppose that there are two shortest vectors, $x \in C$ and $y \in C$. Owing to the convexity, $(x + y)/2 \in C$ and therefore $\|(x + y)/2\|^2 \geq \|x\|^2 = \|y\|^2 = (\|x\|^2 + \|y\|^2)/2$. As a result we get

$$0 \geq \|x\|^2/4 - (x, y)/2 + \|y\|^2/4 = \|x - y\|^2/4.$$

b) Show that the set $C = \{\, x \in H \mid f(x) = 1 \,\}$ is non-empty, closed, and convex.

Solution:
Since f is continuous and $\{1\} \subset \mathbb{R}$ is closed then so is $C = f^{-1}(\{1\})$. Its convexity follows immediately from the linearity of f. Finally, since $f \neq 0$ there is at least one $x_0 \in H$: $f(x_0) \neq 0$. But then $x_0/f(x_0) \in C$ (f is linear).

c) Let $N = \ker f = \{\, x \in H \mid f(x) = 0 \,\}$, and let further $\bar{x}$ be the unique vector of the shortest length in C. Show that $C = \bar{x} + N$.

Solution:
Indeed, $\forall z \in N$ we have $f(\bar{x} + z) = f(\bar{x}) + f(z) = 1$ and therefore $\bar{x} + N \subset C$. On the other hand for any $x \in C$ we have $f(x - \bar{x}) = f(x) - f(\bar{x}) = 1 - 1 = 0$ and therefore $C - \bar{x} \subset N$.

d) Show that $\bar{x} \perp N$. Hint: $\forall x \in N, \delta \in \mathbb{R}$, $\bar{x} + \delta x \in C \implies \|\bar{x} + \delta x\|^2 > \|\bar{x}\|^2$. Consider the limit of $\delta \to 0$.

Solution:

Indeed, for all $x \in N$ and $\delta > 0$ we have $\delta x \in N$ and therefore $\bar{x} + \delta x \in C$. Since $\bar{x}$ is the shortest vector in C we have

$$\|\bar{x} + \delta x\|^2 = \|\bar{x}\|^2 + 2\delta(x, \bar{x}) + \delta^2 \|x\|^2 > \|\bar{x}\|^2.$$

We divide by $\delta > 0$ and let it go to zero to derive the inequality $(\bar{x}, x) \geq 0$, $\forall x \in N$. But for every $x \in N$ we have also $-x \in N$, and therefore $\bar{x} \perp N$.

e) Take any $x \in H \setminus N$. Using the representation of $C = \bar{x} + N$ show that x can be written as $x = f(x)\bar{x} + x_0$ for some $x_0 \in N$. Conclude that $f(x) = (x, \bar{x}/\|\bar{x}\|^2)$. This concludes the existence part of the Riesz representation theorem.

Solution:
Let $x \in H \setminus N$ be arbitrary. Then $f(x) \neq 0$ and $f(x/f(x)) = 1$, thus $x/f(x) \in C = \bar{x} + N$. Therefore, for some $x_0 \in N$ we have $x = f(x)\bar{x} + x_0$. Thus $(x, \bar{x}) = f(x)(\bar{x}, \bar{x}) + (x_0, \bar{x}) = f(x)\|x_0\|^2$, and as a result $f(x) = (x, \bar{x}/\|\bar{x}\|^2)$.
If $x \in N$ then also $0 = f(x) = (x, \bar{x}/\|\bar{x}\|^2) = 0$.
It is probably worth mentioning that $\|\bar{x}\| \neq 0$ since $f(\bar{x}) = 1$ and f is linear.

f) Prove that the Riesz map is an isometry, that is, that $\|f\|_{H'} = \|x_f\|_H$. Show that this also implies the uniqueness of representation x_f in the theorem.

Solution:
Indeed, let $x_f = \bar{x}/\|\bar{x}\|^2 \neq 0$ constructed in the previous point. Then

$$\|f\|_{H'} = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x \neq 0} \frac{(x, x_f)}{\|x\|} = \frac{(x_f, x_f)}{\|x_f\|} = \|x_f\|,$$

where we use Cauchy–Schwarz inequality $(x, x_f) \leq \|x\|\|x_f\|$.
Therefore if $f_1 \neq f_2 \in H'$ correspond to the same $x_f \in H$, then $f_1 - f_2 \mapsto x_f - x_f = 0$. Thus $\|f_1 - f_2\|_{H'} = \|0\|_H = 0$.

2 We will prove a somewhat simpler version of the following statement: a bounded set in a reflexive Banach space is relatively weakly sequentially compact.

More precisely, let $\{x_k\}_{k=1}^\infty$ be a sequence of vectors in a Banach space X which is bounded: $\exists C > 0 : \forall k = 1, 2, \dots, \|x_k\|_X \leq C$. Let further X' be *separable*, that is, let $S = \{f_k\}_{k=1}^\infty$ be an everywhere dense subset in X'. Then from $\{x_k\}$ we can extract a weakly convergent subsequence.

The proof technique is a so-called diagonal process.

a) Let $x_k^{(0)} = x_k$. For each $i = 1, 2, \dots$ show that the sequence $\{f_i(x_k^{(i-1)})\}_{k=1}^\infty$ is bounded in $\mathbb{R}$ and therefore contains a convergent subsequence, which we denote $\{f_i(x_k^{(i)})\}_{k=1}^\infty$.

Solution:
We have the bound $\|f_i(x_k^{(i-1)}) \leq \|f_i\|_{X'}\|x_k^{(i-1)}\|_X \leq C\|f_i\|_{X'}$. Thus the sequence $\{f_i(x_k^{(i-1)})\}_{k=1}^\infty$ is bounded in $\mathbb{R}$ fpr each i and therefore contains a convergent subsequence.

b) Consider now the diagonal subsequence $\{x_k^{(k)}\}_{k=1}^\infty$ of $\{x_k\}_{k=1}^\infty$. Show that $\forall i = 1, 2, \dots$, the sequence $\{f_i(x_k^{(k)})\}_{k=1}^\infty$ converges in $\mathbb{R}$.

Solution:
The diagonal sequence $\{x_k^{(k)}\}_{k=1}^\infty$ (with the exception of the first few terms $\{x_k^{(k)}\}_{k=1}^{i-1}$) is a subsequence of $\{x_k^{(i)}\}_{k=1}^\infty$. Therefore, by construction we have the claimed convergence for every i.

c) Define a function $F : \text{span}(S) \to \mathbb{R}$ as $F(f_i) = \lim_{k\to\infty} f_i(x_k^{(k)})$. Show that F is a linear bounded functional on $\text{span}(S)$ with $\|F\|_{\text{span}(S)'} \le C$.

Solution:
Linearity with respect to f follows from the linearity of lim with respect to the sequence. The boundedness of F is also easy:

$$|f_i(x_k^{(k)})| \le C\|f_i\|_{X'} \quad \Longrightarrow \quad |F(f_i)| \le C\|f_i\|_{X'} \quad \Longrightarrow \quad \|F\|_{X''} \le C.$$

d) Using the previous point and the density of S in X' show that we can extend F to be a linear bounded functional on X' with $\|F\|_{X''} \le C$. Hint: begin by showing that if $\lim_{i\to\infty} \|f_i - f\|_{X'} = 0$ for some $f \in X'$, $\{f_i\}_{i=1}^\infty \subset S$ then the sequence $\{F(f_i)\}_{i=1}^\infty$ is Cauchy.

Solution:
Since $\{f_i\}$ is everywhere dense in X', then every $f \in X'$ can be represented as a limit of some sequence (abuse of notation here) f_i. In particular, f_i is Cauchy, and owing to the boundedness of F (established previously) the sequence $F(f_i)$ is also Cauchy. Since $\mathbb{R}$ is complete, the sequence has a limit, which we call $F(f)$.
Now we must show that this limit is independent from which sequence is used to approximate f. Indeed, if there are two sequences converging to f, say f_i and $\tilde{f}_i$, then the "mixed" sequence $\{g_i\} = \{f_1, \tilde{f}_1, f_2, \tilde{f}_2, \dots\}$ also converges to f. In particular it is Cauchy and then $F(g_i)$ is Cauchy. Thus $F(\lim f_i) = F(\lim \tilde{f}_i)$.
Finally, we need to show linearity and boundedness. This follows immediately from the linearity of F on $\{f_i\}$ and boundedness there as well: if $f = \lim f_i$ in X' then

$$|F(f)| = \lim_i |F(f_i)| \le C \lim_i \|f_i\|_{X'} = C\|f\|_X.$$

e) Use the reflexivity of X to assert the existence of $\bar{x} \in X$ corresponding to $F \in X''$. Show that $\bar{x}$ is the weak limit of $\{x_k^{(k)}\}$.

Solution:
Owing to the reflexivity of X there is $\bar{x} \in X$ such that $\forall f \in X'$ we have $F(f) = f(\bar{x})$. Then, $\forall f \in X'$:

$$\begin{aligned}
|f(\bar{x}) - f(x_k^{(k)})| &= |F(f) - f(x_k^{(k)})| \\
&\le |F(f) - F(f_i)| + |F(f_i) - f_i(x_k^{(k)})| + |f_i(x_k^{(k)}) - f(x_k^{(k)})| \\
&\le C\|f - f_i\|_{X'} + |F(f_i) - f_i(x_k^{(k)})| + C\|f - f_i\|_{X'}.
\end{aligned}$$

We first fix i such that the first and the last terms are smaller than a given $\varepsilon > 0$ (can do this owing to the density of f_i in X'). For this fixed i we know that $F(f_i) = \lim_{k\to\infty} f_i(x_k^{(k)})$, and thus the second term will be smaller than ε for all large k. In other words, $x_k^{(k)} \rightharpoonup \bar{x}$ in X.

3 One of the fundamental theorems of functional analysis, Hahn–Banach separation theorem, states that for any closed convex set C in a Banach space X and a point $\bar{x} \in X \setminus C$ there is $f \in X'$ and $\alpha \in \mathbb{R}$, such that $f(\bar{x}) > \alpha \geq f(x)$, $\forall x \in C$.

Infer from this theorem that if C is a closed convex set and $\{x_k\}_{k=1}^{\infty}$ is a sequence of points from C converging weakly to $\bar{x}$, then $\bar{x} \in C$. (This exercise shows that *convex* closed sets are also closed with respect to the weak convergence. Recall that non-convex closed sets are not necessarily closed with respect to the weak convergence: e.g. a sequence of orthornormal basis vectors in a separable Hilbert space lies on a closed unit sphere but its weak limit 0 lies outside of that closed set.)

Solution:

Suppose that $C \ni x_k \rightharpoonup \bar{x} \in X \setminus C$. We utilize the Hahn–Banach separation theorem; thus for some $f \in X'$ we have $f(\bar{x}) > \alpha \geq f(x_k)$, $\forall k$. Thus $f(\bar{x}) \neq \lim_{k\to\infty} f(x_k)$ and therefore $x_k \not\rightharpoonup \bar{x}$ - contradiction.

4 Let X be a normed space and $f : X \to \mathbb{R}$ be a function. The set epi $f = \{ (x, y) \in X \times \mathbb{R} \mid f(x) \leq y \}$ is called the epigraph of f.

a) Show that f is convex iff its epigraph is a convex set.

Solution:
If f is convex and $(x_1, y_1), (x_2, y_2) \in$ epi f then

$$f(\lambda x_1 + (1-\lambda)x_2) \underbrace{\leq}_{f\text{-convex}} \lambda f(x_1) + (1-\lambda)f(x_2) \underbrace{\leq}_{(x_i,y_i)\in\text{epi } f} \lambda y_1 + (1-\lambda)y_2,$$

for any $\lambda \in (0, 1)$, and therefore $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in$ epi f. Thus the epigraph is convex.
Assume now that the epigraph is convex. Note that for any x_1, x_2 we have $(x_i, f(x_i)) \in$ epi f. Therefore also $(\lambda x_1 + (1-\lambda)x_2, \lambda f(x_1) + (1-\lambda)f(x_2)) \in$ epi f for any $\lambda \in (0, 1)$, and per definition $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$. Therefore, f is convex.

b) A function f is called lower semi-continuous (l.s.c.) at $\bar{x} \in X$ if for any $x_k \to \bar{x}$ we have the inequality $f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k)$. (Somewhat more ackwardly one can say that f is l.s.c. at $\bar{x} \in X$ if for any $x_k \to \bar{x}$ such that $f(x_k)$ converges in $\mathbb{R}$, possibly to $\pm\infty$, we have the inequality $f(\bar{x}) \leq \lim_{k\to\infty} f(x_k)$.) Show that f is l.s.c. at every $x \in X$ iff its epigraph is a closed set in $X \times \mathbb{R}$. (The norm on $X \times \mathbb{R}$ can for example be defined as $\|(x, y)\| = \|x\|_X + |y|$, that is, $(x_k, y_k) \to (\bar{x}, \bar{y})$ iff $x_k \to \bar{x}$ and $y_k \to \bar{y}$.)

Solution:
Suppose that for some $\bar{x}$ we have $x_k \to \bar{x}$ but $f(\bar{x}) > \lim_{N\to\infty} \inf_{k\geq N} f(x_k)$. Then there is $\varepsilon > 0$ and $\hat{N}$: for all $N \geq \hat{N}$ we have $f(\bar{x}) - \varepsilon > \inf_{k\geq N} f(x_k)$.

Take any x_{k_1}: $f(x_{k_1}) < \inf_{k\geq\hat{N}} f(x_k) + \varepsilon/2$. Then select $x_{k_{i+1}}$: $f(x_{k_{i+1}}) < \inf_{k\geq k_i+1} f(x_k) + \varepsilon/2$.

Thus we can take epi $f \ni (x_{k_i}, f(\bar{x}) - \varepsilon/4) \to (\bar{x}, f(\bar{x}) - \varepsilon/4) \notin$ epi f, showing that the epigraph is not a closed set.

Assume now that the function f is l.s.c. and take any convergent sequence epi $f \ni (x_k, y_k) \to (\bar{x}, \bar{y})$. Then

$$f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k) \leq \liminf_{k\to\infty} y_k = \lim_{k\to\infty} y_k = \bar{y},$$

and $(\bar{x}, \bar{y}) \in$ epi f. Thus epi f is a closed set.

c) Combine the results in [3], [4] a)–b) to show that a convex l.s.c. function is also lower semicontinuous with respect to the weak convergence on X. That is, $x_k \rightharpoonup \bar{x} \implies f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k)$.

Solution:

We have established that f-closed and convex $\iff$ epi f is closed and convex. Thus epi f is also closed with respect to the weak convergence in $X \times \mathbb{R}$.

A linear bounded functional on $X \times \mathbb{R}$ (with respect to the norm $\|(x, y)\| = \|x\|_X + |y|$) must be linear and bounded with respect to each individual argument. Therefore they must have the form $(x, y) \mapsto f(x) + \alpha y$, $f \in X'$, $\alpha \in \mathbb{R}$. Weak convergence with respect to such functionals means weak convergence of x-components and convergence of y-components in $\mathbb{R}$.

Finally, one proceeds as in the first part of the proof of point b) while replacing the strong convergence of x-components with weak convergence.