TMA4183 Opt. II
Spring 2016

Exercise set 2

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Reading material: Sections 2.1–2.4 from [Tröltsch].

1 On monday we have proved the following result: let C be a non-empty closed and convex subset of a Hilbert space H. Then there is $\bar{x} \in C$ with the smallest length, that is, $\|\bar{x}\| = \min_{x\in C} \|x\|$. Now we will use this result to prove Riesz representation theorem: for any $f \in H'$ there is a unique $x_f \in H$ such that $\forall x \in H : f(x) = (x, x_f)$ and $\|f\|_{H'} = \|x_f\|_H$.

Since this result is trivial for $f \equiv 0$, we will assume that $\|f\| \neq 0$.

a) Let C be a non-empty closed convex subset of H. Show that there is only one vector of shortest length in it. Hint: $2(x, y) \leq \|x\|^2 + \|y\|^2$ with equality iff $\|x - y\|^2 = 0$.

b) Show that the set $C = \{ x \in H \mid f(x) = 1 \}$ is non-empty, closed, and convex.

c) Let $N = \ker f = \{ x \in H \mid f(x) = 0 \}$, and let further $\bar{x}$ be the unique vector of the shortest length in C. Show that $C = \bar{x} + N$.

d) Show that $\bar{x} \perp N$. Hint: $\forall x \in N, \delta \in \mathbb{R}$, $\bar{x} + \delta x \in C \implies \|\bar{x} + \delta x\|^2 > \|\bar{x}\|^2$. Consider the limit of $\delta \to 0$.

e) Take any $x \in H \setminus N$. Using the representation of $C = \bar{x} + N$ show that x can be written as $x = f(x)\bar{x} + x_0$ for some $x_0 \in N$. Conclude that $f(x) = (x, \bar{x}/\|\bar{x}\|^2)$. This concludes the existence part of the Riesz representation theorem.

f) Prove that the Riesz map is an isometry, that is, that $\|f\|_{H'} = \|x_f\|_H$. Show that this also implies the uniqueness of representationx_f in the theorem.

2 We will prove a somewhat simpler version of the following statement: a bounded set in a reflexive Banach space is relatively weakly sequentially compact.

More precisely, let $\{x_k\}_{k=1}^{\infty}$ be a sequence of vectors in a Banach space X which is bounded: $\exists C > 0 : \forall k = 1, 2, \dots, \|x_k\|_X \leq C$. Let further X' be *separable*, that is, let $S = \{f_k\}_{k=1}^{\infty}$ be an everywhere dense subset in X'. Then from $\{x_k\}$ we can extract a weakly convergent subsequence.

The proof technique is a so-called diagonal process.

a) Let $x_k^{(0)} = x_k$. For each $i = 1, 2, \dots$ show that the sequence $\{f_i(x_k^{(i-1)})\}_{k=1}^{\infty}$ is bounded in $\mathbb{R}$ and therefore contains a convergent subsequence, which we denote $\{f_i(x_k^{(i)})\}_{k=1}^{\infty}$.

b) Consider now the diagonal subsequence $\{x_k^{(k)}\}_{k=1}^\infty$ of $\{x_k\}_{k=1}^\infty$. Show that $\forall i = 1, 2, \dots$, the sequence $\{f_i(x_k^{(k)})\}_{k=1}^\infty$ converges in $\mathbb{R}$.

c) Define a function $F : \mathrm{span}(S) \to \mathbb{R}$ as $F(f_i) = \lim_{k\to\infty} f_i(x_k^{(k)})$. Show that F is a linear bounded functional on $\mathrm{span}(S)$ with $\|F\|_{\mathrm{span}(S)'} \le C$.

d) Using the previous point and the density of S in X' show that we can extend F to be a linear bounded functional on X' with $\|F\|_{X''} \le C$. Hint: begin by showing that if $\lim_{i\to\infty} \|f_i - f\|_{X'} = 0$ for some $f \in X'$, $\{f_i\}_{i=1}^\infty \subset S$ then the sequence $\{F(f_i)\}_{i=1}^\infty$ is Cauchy.

e) Use the reflexivity of X to assert the existence of $\bar{x} \in X$ corresponding to $F \in X''$. Show that $\bar{x}$ is the weak limit of $\{x_k^{(k)}\}$.

3 One of the fundamental theorems of functional analysis, Hahn–Banach separation theorem, states that for any closed convex set C in a Banach space X and a point $\bar{x} \in X \setminus C$ there is $f \in X'$ and $\alpha \in \mathbb{R}$, such that $f(\bar{x}) > \alpha \ge f(x)$, $\forall x \in C$.

Infer from this theorem that if C is a closed convex set and $\{x_k\}_{k=1}^\infty$ is a sequence of points from C converging weakly to $\bar{x}$, then $\bar{x} \in C$. (This exercise shows that *convex* closed sets are also closed with respect to the weak convergence. Recall that non-convex closed sets are not necessarily closed with respect to the weak convergence: e.g. a sequence of orthornormal basis vectors in a separable Hilbert space lies on a closed unit sphere but its weak limit 0 lies outside of that closed set.)

4 Let X be a normed space and $f : X \to \mathbb{R}$ be a function. The set $\mathrm{epi}\, f = \{\,(x, y) \in X \times \mathbb{R} \mid f(x) \le y\,\}$ is called the epigraph of f.

a) Show that f is convex iff its epigraph is a convex set.

b) A function f is called lower semi-continuous (l.s.c.) at $\bar{x} \in X$ if for any $x_k \to \bar{x}$ we have the inequality $f(\bar{x}) \le \liminf_{k\to\infty} f(x_k)$. (Somewhat more ackwardly one can say that f is l.s.c. at $\bar{x} \in X$ if for any $x_k \to \bar{x}$ such that $f(x_k)$ converges in $\mathbb{R}$, possibly to $\pm\infty$, we have the inequality $f(\bar{x}) \le \lim_{k\to\infty} f(x_k)$.) Show that f is l.s.c. at every $x \in X$ iff its epigraph is a closed set in $X \times \mathbb{R}$. (The norm on $X \times \mathbb{R}$ can for example be defined as $\|(x, y)\| = \|x\|_X + |y|$, that is, $(x_k, y_k) \to (\bar{x}, \bar{y})$ iff $x_k \to \bar{x}$ and $y_k \to \bar{y}$.)

c) Combine the results in 3, 4 a)–b) to show that a convex l.s.c. function is also lower semicontinuous with respect to the weak convergence on X. That is, $x_k \rightharpoonup \bar{x} \implies f(\bar{x}) \le \liminf_{k\to\infty} f(x_k)$.