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TMA4183 Opt. II
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Exercise set 1

Reading material: Chapter 1 & Section 2.1–2.2 from [Tröltsch].

1 We consider a(n artificial) finite-dimensional optimal control problem for $y \in \mathbb{R}^2$ with a control parameter $u \in \mathbb{R}$.

The state equation is:

$$\begin{aligned} y_1 + y_2 &= u, \\ y_2 &= 2u, \end{aligned} \tag{1}$$

and the const functional is

$$J(y,u) = \frac{1}{2}[(y_1 - 1)^2 + (y_2 - 2)^2] + \frac{\lambda}{2}u^2, \tag{2}$$

where $\lambda > 0$.

a) Derive the explicit expressions for the reduced cost functional and its gradient.

Solution: The control-to-state operator $y = Su$ is obtained by solving the state equations yeilding $S = [-1, 2]^T$. The reduced cost function and its gradient are:

$$\begin{aligned} f(u) &= J(Su, u) = \frac{5+\lambda}{2}u^2 - 3u + \frac{5}{2}, \\ f'(u) &= (5+\lambda)u - 3. \end{aligned}$$

b) Formulate the adjoint problem and compute the reduced gradient with the help of the adjoint state.

Solution: The state equation in the matrix-vector form can be statet as

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{=:A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{=:B} u.$$

He adjoint system is then $A^T p = \nabla_y J$, or

$$\begin{aligned} p_1 &= y_1 - 1, \\ p_1 + p_2 &= y_2 - 2, \end{aligned}$$

thus $p_2 = -y_1 + y_2 - 1$. Finally, the reduced gradient is

$$\begin{aligned} f'(u) &= B^T p + \nabla_u J = 1(y_1 - 1) + 2(-y_1 + y_2 - 1) + \lambda u \\ &= -u - 1 + 2(u + 2u - 1) + \lambda u = (5+\lambda)u - 3. \end{aligned}$$

c) Assuming $U_{\text{ad}} = \mathbb{R}$ state the first order necessary optimality conditions for this problem.

Solution: In the absense of restrictions on the control the first order necessary optimality conditions are

$$\begin{aligned} Ay &= Bu \\ A^T p &= \nabla_y J \\ \underbrace{B^T p + \nabla_u J}_{=f'(u)} &= 0. \end{aligned}$$

These can even be solved, namely $u = 3/(5+\lambda)$ etc.

2 Consider the definition of a domain of class $C^{k,1}$ on p. 26, Section 2.2 in [Tr]. Describe in detail the objects (cubes, functions h_i, etc) appearing in the definition when (a) Ω = unit square in $\mathbb{R}^2$; (b) Ω = unit ball in $\mathbb{R}^2$.

It is probably easiest to subdivide the boundary into four parts in both cases.

Solution:

For the unit circle one can for example decompose the boundary into four overlapping neighbourhoods, corresponding to the parts (in polar coordinates) $\pi/6 < \phi < \pi - \pi/6$; $\pi/2 + \pi/6 < \phi < 3\pi/2 - \pi/6$; $\pi + \pi/6 < \phi < 2\pi - \pi/6$; $3\pi/2 + \pi/6 < \phi < 2\pi + \pi/2 - \pi/6$. For the first part, the unit circle (near the boundary) is $-y_2 > -h_1(y_1) = -\sqrt{1-y_1^2}$, inside the cube (interval) $-\sqrt{3}/2 < y_1 < \sqrt{3}/2$, where the local coordinates are simply $y_i = x_i$. in this way $h_1 \in C^{k,1}(-\sqrt{3}/2, \sqrt{3}/2)$ for all k.

For the third part of the boundary we can take the same coordinate system but we need a different inequality: $y_2 > h_3(y_1) = -\sqrt{1-y_1^2}$, inside the cube (interval) $-\sqrt{3}/2 < y_1 < \sqrt{3}/2$.

Similarly for the other 2 cases.

In the case of a unit square $|x_1| < 1$, $|x_2| < 1$, we split the boundary into four open overlapping neighbourhoods centered around the corners. For the right/bottom corner we can use the coordinate system $y_1 = x_1 + x_2$, $y_2 = x_2 - x_1$ and $h(y_1) = |y_1|$ inside the cube $-2 < y_1 < 2$. Then h is only Lipschitz (i.e., $k = 0$).

Similar arguments for the other three corners.

3 **a)** Show that the weak derivative of $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = |x|$ is

$$g(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Note that it is not necessary to define g at 0, which has measure 0. Thus $f \in W^{1,p}(a,b)$ for an arbitrary $a < b$ and arbitrary $1 \le p \le \infty$.

Solution: Indeed for arbitrary $a < 0 < b$ and an arbitrary $\phi \in C_0^\infty(a,b)$ we have

$$\int_a^b |x|\phi'(x)\,\mathrm{d}x = -\int_a^0 x\phi'(x) + \int_0^b x\phi'(x) = \int_a^0 \phi(x) - \int_0^b \phi(x) = -1\int_a^b g(x)\phi(x)\,\mathrm{d}x,$$

where the second inequality is obtained by integrating by parts and noting that $\phi(a) = \phi(b) = 0$ and $x|_0 = 0$. Thus g is the weak derivative of f.

b) Show that f in the previous example is *not* twice weakly differentiable. (This example shows than not all functions are weakly differentiable.)

Hint: take an arbitrary $\phi \in C_0^\infty(\mathbb{R})$, such that $\phi(0) \neq 0$, and put $\phi_k(x) = \phi(kx)$. Assume that equality (2.1) in the book holds for some integrable function (=potential weak derivative), and consider the limit of both sides of the equality for $k \to \infty$. Use the dominated Lebesgue convergence theorem to switch from the pointwise convergence of ϕ_k to the convergence of the integrals.

Solution: Assume that the weak second derivative of f exists and equals h, that is, for any $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\int f(x)\phi''(x) = \int h(x)\phi(x).$$

Note that if $\operatorname{supp}\phi \subset [-N,N]$ then also $\operatorname{supp}\phi' \subset [-N,N]$ and in particular $\phi' \in C_0^\infty(\mathbb{R})$. Therefore, owing to (a) we get

$$\int f(x)\phi''(x) = -\int g(x)\phi'(x),$$

thus the weak second derivative of f is the weak first derivative of g.

Let us now assume that $\phi(0) \neq 0$ and construct $\phi_k(x) = \phi(kx)$. Then $phi_k(0) = \phi(0) \neq 0$ and $\operatorname{supp}\phi_k \subset [-N/k, N/k]$. In particular, for any $x \neq 0$ we have $\phi_k(x) = \phi(kx) = 0$ for $k > N/|x|$. Thus $\phi_k(x) \to 0$ as $k \to \infty$, pointwise, almost everywhere (in this case except at $x = 0$).

Finally, we compute

$$-\int g(x)\phi_k'(x) = \int_{N/k}^0 \phi_k'(x) - \int_0^{N/k} \phi_k'(x) = \phi_k(0) + \phi_k(0) = 2\phi_k(0) = 2\phi(0) \neq 0.$$

On the other hand we know that $|\phi_k(x)h(x)| \leq \|\phi_x\|_{L^\infty(\mathbb{R})}|h(x)|$, and $|h(x)|$ is a Lebesgue integrable function on $[-N,N]$ (by our assumption). Therefore Lebesgue dominated convergence theorem applies and

$$\int_{-N}^N h(x)\phi_k(x) \to \int_{-N}^N h(x)\cdot 0 = 0 \neq 2\phi_k(0) = -\int_{-N}^N g(x)\phi_k'(x)$$

which is a contradiction.

c) Cet B be an open unit ball in $\mathbb{R}^n$, and define $f(x) = \|x\|^{-\gamma}$, $\gamma > 0$. Note that the function "blows up" at 0 but is in $C^\infty(B \setminus \{0\})$. Let $g(x) = \nabla f(x)$ for $x \neq 0$. Derive the conditions on γ to show that g is the weak derivative of f in B. This example shows that some discontinuous/unbounded functions are weakly differentiable.

Hint: fix an arbitrary $\phi \in C_0^\infty(B)$. Then derive bounds on γ under which both f and g are integrable in B, and the following holds:

$$\int_B f D_i\phi = \int_{B\setminus\varepsilon B} f D_i\phi + \underbrace{\int_{\varepsilon B} f D_i\phi}_{\to 0, \text{as } \varepsilon\to 0},$$

$$\int_B g_i\phi = \int_{B\setminus\varepsilon B} g_i\phi + \underbrace{\int_{\varepsilon B} g_i\phi}_{\to 0, \text{as } \varepsilon\to 0},$$

$$\left|\int_{B\setminus\varepsilon B} f D_i\phi + \int_{B\setminus\varepsilon B} g_i\phi\right| = \underbrace{\int_{\partial\varepsilon B} f\phi\nu_i}_{\to 0, \text{as } \varepsilon\to 0},$$

where ν is the unit normal to $B\setminus\varepsilon B$. Use spherical coordinates to estimate the "small" integrals.

Solution:

So the main problem is to remove the singularity at 0, because the function is differentiable elsewhere. Indeed, let $g(x) = \nabla\|x\|^{-\gamma} = -\gamma\|x\|^{-\gamma-1}\nabla\|x\| = -\gamma\|x\|^{-\gamma-2}x$ for $x\neq 0$. In particular $|g_i(x)| \le \|g(x)\| = \gamma\|x\|^{-\gamma-1}$.

Let us fix an arbitrary $\phi\in C_0^\infty(B)$, and let us estimate the integrals around the singularity. We do this by using hyperspherical coordinates, and by C_n we denote the surphace of the unit sphere in $\mathbb{R}^n$.

$$\begin{aligned}\left|\int_{\varepsilon B} f D_i\phi\right| &\le \|D_i\phi\|_{L^\infty(B)}\int_{\varepsilon B}|f| = \|D_i\phi\|_{L^\infty(B)} C_n\int_0^\varepsilon r^{n-1}r^{-\gamma}\\ &= \|D_i\phi\|_{L^\infty(B)} C_n\left[\frac{r^{n-\gamma}}{n-\gamma}\right]_{r=0}^{\varepsilon}\to 0\end{aligned}$$

as $\varepsilon\to 0$ for all $\gamma < n$.

Similarly

$$\begin{aligned}\left|\int_{\varepsilon B} g_i\phi\right| &\le \gamma\|\phi\|_{L^\infty(B)}\int_{\varepsilon B}\|x\|^{-\gamma-1} = \gamma\|\phi\|_{L^\infty(B)} C_n\int_0^\varepsilon r^{n-1}r^{-\gamma-1}\\ &= \|\phi\|_{L^\infty(B)}\gamma C_n\left[\frac{r^{n-\gamma-1}}{n-\gamma-1}\right]_{r=0}^{\varepsilon}\to 0\end{aligned}$$

for all $\gamma < n-1$.

For the surface integral we get

$$\left|\int_{\partial\varepsilon B} f\phi\nu_i\right| \le \|\phi\|_{L^\infty(B)}\int_{\partial\varepsilon B}|f| = \|D_i\phi\|_{L^\infty(B)} C_n\varepsilon^{n-1}\varepsilon^{-\gamma}\to 0$$

for all $\gamma < n-1$.

As a result we can write

$$\begin{aligned}\int_B fD_i\phi + \int_B g_i\phi &= \int_{B\setminus\varepsilon B} fD_i\phi + \int_{B\setminus\varepsilon B} g_i\phi + \int_{\varepsilon B} fD_i\phi + \int_{\varepsilon B} g_i\phi\\ &= \int_{B\setminus\varepsilon B} D_i[f\phi] + \int_{\varepsilon B} fD_i\phi + \int_{\varepsilon B} g_i\phi\\ &= -\int_{\partial\varepsilon B} f\phi\nu_i + \underbrace{\int_{\partial B} f\phi\nu_i}_{=0 \text{ since } \phi\in C_0^\infty(B)} + \int_{\varepsilon B} fD_i\phi + \int_{\varepsilon B} g_i\phi \to 0.\end{aligned}$$

Since the left hand side is independent from ε we must have the equality

$$\int_B f D_i \phi = -\int_B g_i \phi,$$

for any $\phi \in C_0^\infty(B)$, or that g is the weak derivative of f as long as $\gamma < n - 1$. This does not exclude unbounded functions for $n > 1$!

Of course if further regularity is required, for example that both f and g are square integrable, further restrictions on γ arise.