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TMA4183 Opt. II
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Exercise set 1

Reading material: Chapter 1 & Section 2.1–2.2 from [Tröltsch].

1 We consider a(n artificial) finite-dimensional optimal control problem for $y \in \mathbb{R}^2$ with a control parameter $u \in \mathbb{R}$.

The state equation is:

$$\begin{aligned} y_1 + y_2 &= u, \\ y_2 &= 2u, \end{aligned} \tag{1}$$

and the const functional is

$$J(y,u) = \frac{1}{2}[(y_1 - 1)^2 + (y_2 - 2)^2] + \frac{\lambda}{2}u^2, \tag{2}$$

where $\lambda > 0$.

a) Derive the explicit expressions for the reduced cost functional and its gradient.

b) Formulate the adjoint problem and compute the reduced gradient with the help of the adjoint state.

c) Assuming $U_{\text{ad}} = \mathbb{R}$ state the first order necessary optimality conditions for this problem.

2 Consider the definition of a domain of class $C^{k,1}$ on p. 26, Section 2.2 in [Tr]. Describe in detail the objects (cubes, functions h_i, etc) appearing in the definition when (a) Ω = unit square in $\mathbb{R}^2$; (b) Ω = unit ball in $\mathbb{R}^2$.

It is probably easiest to subdivide the boundary into four parts in both cases.

3 **a)** Show that the weak derivative of $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = |x|$ is

$$g(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Note that it is not necessary to define g at 0, which has measure 0. Thus $f \in W^{1,p}(a,b)$ for an arbitrary $a < b$ and arbitrary $1 \le p \le \infty$.

b) Show that f in the previous example is *not* twice weakly differentiable. (This example shows than not all functions are weakly differentiable.)

Hint: take an arbitrary $\phi \in C_0^\infty(\mathbb{R})$, such that $\phi(0) \neq 0$, and put $\phi_k(x) = \phi(kx)$. Assume that equality (2.1) in the book holds for some integrable function (=potential weak derivative), and consider the limit of both sides of the equality for $k \to \infty$. Use the dominated Lebesgue convergence theorem to switch from the pointwise convergence of ϕ_k to the convergence of the integrals.

c) Cet B be an open unit ball in $\mathbb{R}^n$, and define $f(x) = \|x\|^{-\gamma}$, $\gamma > 0$. Note that the function "blows up" at 0 but is in $C^\infty(B \setminus \{0\})$. Let $g(x) = \nabla f(x)$ for $x \neq 0$. Derive the conditions on γ to show that g is the weak derivative of f in B. This example shows that some discontinuous/unbounded functions are weakly differentiable.

Hint: fix an arbitrary $\phi \in C_0^\infty(B)$. Then derive bounds on γ under which both f and g are integrable in B, and the following holds:

$$\int_B f D_i\phi = \int_{B\setminus\varepsilon B} f D_i\phi + \underbrace{\int_{\varepsilon B} f D_i\phi}_{\to 0, \text{as } \varepsilon \to 0},$$

$$\int_B g_i\phi = \int_{B\setminus\varepsilon B} g_i\phi + \underbrace{\int_{\varepsilon B} g_i\phi}_{\to 0, \text{as } \varepsilon \to 0},$$

$$\left| \int_{B\setminus\varepsilon B} f D_i\phi + \int_{B\setminus\varepsilon B} g_i\phi = \underbrace{\int_{\partial\varepsilon B} f\phi\nu_i}_{\to 0, \text{as } \varepsilon \to 0},\right.$$

where ν is the unit normal to $B \setminus \varepsilon B$. Use spherical coordinates to estimate the "small" integrals.