Optimisation I Formula Sheet 2026

This sheet will be provided together with the exam. It is meant to guide your studies. It is not an exhaustive list of the course or the examination curriculum.

Unconstrained optimisation

Taylor Expansion

$$f(x+p) = f(x) + p^\top \nabla f(x) + \frac{1}{2} p^\top \nabla^2 f(x) p + \mathcal{O}(\|p\|^3)$$

- First-order necessary conditions, second-order necessary and sufficient conditions, global minima and convexity.

Line Search Methods

Line Search

$$x_{k+1} = x_k + \alpha_k p_k$$

where $\alpha_k > 0$ is the step length and p_k is the search direction.

Descent Direction: The angle θ between p_k and the steepest descent direction must satisfy:

$$\cos\theta = \frac{-\nabla f(x_k)^\top p_k}{\|\nabla f(x_k)\| \|p_k\|} > 0$$

Common Search Directions

- **Steepest Descent:** $p_k = -\nabla f(x_k)$
- **Newton's Method:** $p_k = -[\nabla^2 f(x_k)]^{-1} \nabla f(x_k)$
- **Quasi-Newton:** $p_k = -B_k^{-1} \nabla f(x_k)$ where $B_k \approx \nabla^2 f(x_k)$.

Hessian update formula

- **DFP**

  $$B_{k+1} = (I_d - \rho_k y_k s_k^T) B_k (I_d - \rho_k s_k y_k^T) + \rho_k y_k y_k^T,$$

  with $\rho_k = (y_k^T s_k)^{-1}$.

- **SR1**

  $$B_{k+1} = B_k + \frac{(y_k - B_k s_k)(y_k - B_k s_k)^T}{(y_k - B_k s_k)^T s_k}$$

Conjugate gradient Conjugate directions, algorithms, theory.

- **Linear case**

  $$\beta_{k+1} = \frac{r_{k+1}^T A p_k}{p_k^T A p_k}$$

- **Nonlinear case**
  - Fletcher-Reeves (FR)

    $$\beta_{k+1} = \frac{\nabla f_{k+1}^T \nabla f_{k+1}}{\nabla f_k^T \nabla f_k}$$

  - Polak-Ribère (PR)

    $$\beta_{k+1} = \frac{\nabla f_{k+1}^T (\nabla f_{k+1} - \nabla f_k)}{\|\nabla f_k\|^2}$$

Trust Region Methods

- Subproblem (formulation and solving). Simplified approximations (Cauchy point, Dogleg, 2D subspace minimisation). Iteratively solving the subproblem.

Quasi-Newton Methods

These methods approximate the Newton step without evaluating the exact Hessian ($B_k \approx \nabla^2 f(x_k)$).

Secant Equation $B_{k+1} s_k = y_k$ where the displacements are defined as

$$s_k = x_{k+1} - x_k, \quad y_k = \nabla f_{k+1} - \nabla f_k.$$

For B_{k+1} to be positive definite, the curvature condition $s_k^\top y_k > 0$ must hold.

Constrained Optimisation

Critical cone $\mathcal{C}(x^\star, \lambda^*) = \{w \in \mathcal{F}(x^\star) \colon \nabla c_i(x^\star)^T w = 0 \text{ for all } i \in \mathcal{A}(x^\star) \cap \mathcal{I} \text{ with } \lambda_i^\star > 0\}$

Dual to convex problem $\min_x f(x); c(x) \geq 0$ **is**

$$\max_\lambda q(\lambda) = \max_\lambda \inf_x \mathcal{L}(x, \lambda); \quad \lambda \geq 0.$$

Slack-formulation dual for $\min_x c^T x; Ax = b, x \geq 0$**:**

$$\max b^T \lambda; \qquad A^T \lambda + s = c, \quad s \geq 0.$$

- Feasible point, Tangent cone $T_\Omega(x^*)$, linear feasible directions $\mathcal{F}(x)$, first-order conditions (KKT), CQ and LICQ and linear constraints, second-order conditions.
- Normal cone and geometric necessary condition for a minimiser, Lagrangian multipliers as sensitivity.
- Dual problem and weak/strong duality. Wolfe dual.
- Linear programs, slack variables, polytope, concept of basic feasible point and relation to geometry. Idea of the simplex method.

Modified Lagrangian Methods

Transform constrained problems into unconstrained ones by penalising constraint violations:

- Penalty and augmented Lagrangian.