Problem 1.

Consider the unconstrained optimisation problem

$$\min_{x,y} f(x, y),$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x, y) = x^4 + 2x^2y + y^2 - 4xy - 4x + 2y.$$

a) Find all local and global minimisers of f.
(10 points)

b) Determine whether the function f is convex.
(5 points)

c) Perform one step of the gradient descent method with backtracking Armijo line search starting from the point $(x, y) = (0, 0)$. Use the parameters $c_1 = 0.1$ (sufficient decrease parameter), $\rho = 0.25$ (contraction factor), and $\alpha_0 = 1$ (initial step length).
(10 points)

Problem 2.

Consider the constrained optimisation problem

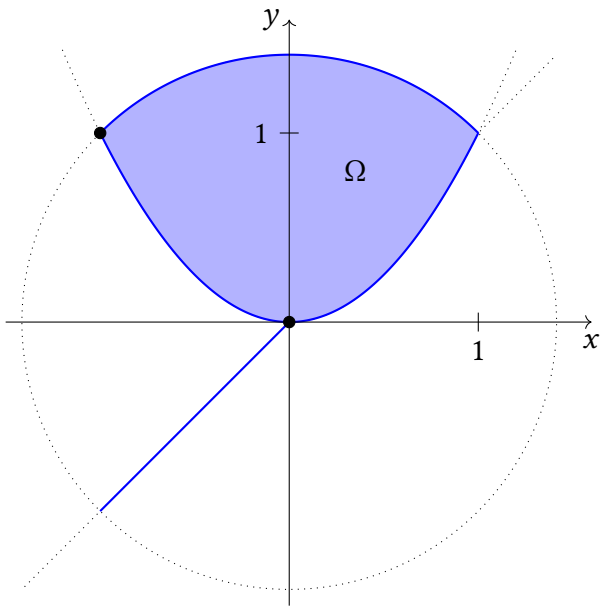
$$\min_{x,y} y^2 - (x - 1)^2 \qquad \text{s.t. } 4y \geq x^2.$$

Use the 2nd order optimality conditions for constrained optimisation to determine whether the point $(x, y) = (2, 1)$ is a local solution of this problem.
(15 points)

Problem 3.

The set $\Omega \subset \mathbb{R}^2$ (see the sketch below) is given by the inequalities

$$(y - x^2)(x - y)^2 \geq 0 \qquad \text{and} \qquad 2 - x^2 - y^2 \geq 0.$$



a) Determine for each point in Ω whether the linear independence constraint qualification (LICQ) holds.
(5 points)

b) Determine the tangent cone $T_\Omega(x, y)$ and the cone of linearised feasible directions $\mathcal{F}(x, y)$ to the set Ω in the points $(x, y) = (-1, 1)$ and $(x, y) = (0, 0)$. (The points are marked in the sketch.)
(10 points)

Problem 4.

Let $A \in \mathbb{R}^{d\times d}$ be symmetric and positive definite, let $c \in \mathbb{R}^d$, and let $b \in \mathbb{R}^d_{>0}$. We want to solve the constrained optimisation problem

$$\min_{x\in\mathbb{R}^d} \frac{1}{2}\langle x, Ax\rangle - \langle x, c\rangle \qquad \text{s.t. } 0 \leq x \leq b.$$

Here the inequalities $0 \leq x \leq b$ are to be interpreted componentwise. Propose an algorithm for the numerical solution of this problem under the assumption that the dimension d is large (e.g. $d \approx 10^4$).
(10 points)

Problem 5.

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be continuously differentiable, let $x \in \mathbb{R}^d$, let $0 < c_1 < 1$, and let $p \in \mathbb{R}^d$ be a descent direction for f at the point x. Show that there exists some $\hat{\alpha} > 0$ such that the Armijo condition

$$f(x + \alpha p) \leq f(x) + c_1 \alpha \langle \nabla f(x), p \rangle$$

holds for all $0 < \alpha < \hat{\alpha}$.
(10 points)

Problem 6.

Let $f, g\colon \mathbb{R}^d \to \mathbb{R}$ be functions and let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. We consider the multicriteria optimisation problems

$$\min_{x \in \mathbb{R}^d} \big(f(x), g(x)\big) \tag{1}$$

and

$$\min_{x \in \mathbb{R}^d} \big(\varphi(f(x)), \varphi(g(x))\big). \tag{2}$$

Show that the problems (1) and (2) have precisely the same solutions.
(10 points)

Problem 7.

Let $f, g\colon \mathbb{R}^d \to \mathbb{R}$ be convex functions. Define the function $h\colon \mathbb{R}^d \to \mathbb{R}$ by

$$h(x) := \inf_{z \in \mathbb{R}^d} \big(f(x - z) + g(z)\big).$$

a) Show that h is convex.
(5 points)

b) Show that the Legendre–Fenchel transform h^* of h satisfies

$$h^*(y) = f^*(y) + g^*(y)$$

for every $y \in \mathbb{R}^d$.
(5 points)

c) Assume now that $f = g$. Show that in this case we have that $h(x) = 2f(x/2)$ for every $x \in \mathbb{R}^d$. That is, for convex $f\colon \mathbb{R}^d \to \mathbb{R}$ we have for every $x \in \mathbb{R}^d$ that

$$\inf_{z \in \mathbb{R}^d} \big(f(x - z) + f(z)\big) = 2f\left(\frac{x}{2}\right).$$

(5 points)