Problem 1.

Consider the unconstrained optimisation problem

$$\min_{x,y} f(x, y), \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x, y) = e^{y^2} + 2x^2 + xy + 4x - \frac{1}{2}y^2 + y.$$

a) Find all local and global minimisers of f.
(10 points)

b) Determine whether the function f is convex.
(5 points)

c) Assume that we want to use the gradient descent method with backtracking Armijo line search for the minimisation of (1). Does the resulting iteration converge for all initialisations to a local or global solution? Briefly explain your reasoning.
(5 points)

d) Perform one step of the gradient descent method with backtracking Armijo line search starting from the point $(x, y) = (0, 0)$. Use the parameters $\hat{\alpha} = 1$ (initial step length), $c_1 = 0.1$ (sufficient decrease parameter) and $\rho = 0.25$ (contraction factor).
(10 points)

Problem 2.

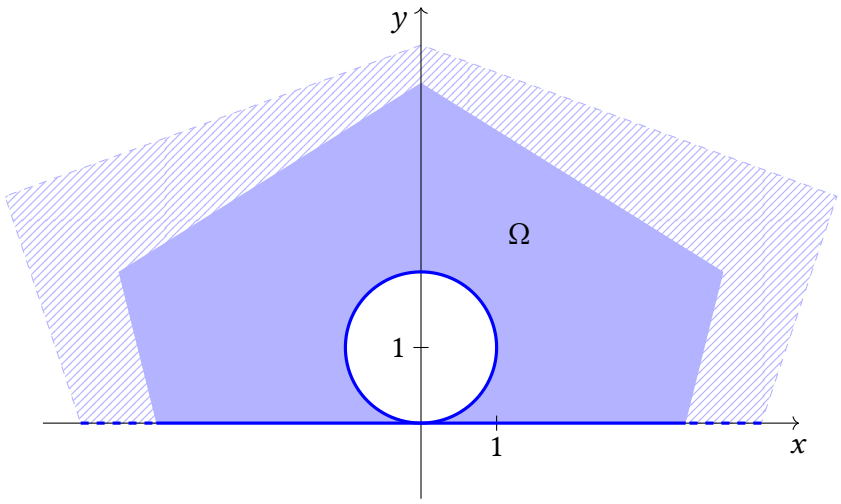
Consider the optimisation problem

$$\min_{x,y} 2x^2 - x^2 y - 3y^3 + 4y^2 + 2y$$

subject to the constraint $(x, y) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is given by the inequalities

$$x^2 + (y-1)^2 \geq 1 \quad \text{and} \quad y \geq 0.$$

Here is a sketch of the set Ω (the set continues infinitely in all directions of the upper half-plane):



a) Determine for each point in Ω whether the LICQ holds.
(5 points)

b) Determine the tangent cone $T_\Omega(x, y)$ and the cone of linearised feasible directions $\mathcal{F}(x, y)$ to the set Ω in the points $(x, y) = (0, 2)$ and $(x, y) = (0, 0)$.
(10 points)

c) Find all KKT points and all local minimisers for this optimisation problem. Use the second order necessary / sufficient conditions whenever applicable. Which of the local minimisers are global minimisers?
(15 points)

Problem 3.

We consider the constrained optimisation problem

$$\min_x f(x) \qquad \text{s.t. } c(x) \geq 0, \tag{2}$$

where $f\colon \mathbb{R}^d \to \mathbb{R}$ is twice continuously differentiable, and $c\colon \mathbb{R}^d \to \mathbb{R}$ is concave and once continuously differentiable. Assume that x^* is a KKT point of (2) and that the Hessian $H_f(x^*)$ is positive definite. Show that x^* is a local solution of (2).
(10 points)

Hint: Use a second order Taylor series expansion of f! Be aware, though, that the function c is not *assumed to be twice differentiable, so the Hessian H_c cannot be assumed to exist.*

Problem 4.

Assume that we want to solve an optimisation problem $\min_x f(x)$, where $f\colon \mathbb{R}^d \to \mathbb{R}$ with a (non-linear) CG method with *exact line search.* Show that the Polak–Ribière method and the Hestenes–Stiefel method yield identical steps, if they are both initialised with $p_0 = -\nabla f(x_0)$.
(10 points)

Problem 5.

Define the function $f\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$,

$$f(x, y) = \sum_{i=1}^{d} \exp(x_i),$$

let $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$. We consider the constrained optimisation problem

$$\min_{(x,y)\in\mathbb{R}^d\times\mathbb{R}} f(x, y) \qquad \text{s.t. } Ax \leq yb. \tag{3}$$

(Here the variable $y \in \mathbb{R}$ only appears in the constraint; the function f only depends on x.)

a) Compute the Lagrangian dual of the optimisation problem (3).
(10 points)

b) Show that the dual problem has a solution for all $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$.
(5 points)

c) Find an example of a matrix $A \in \mathbb{R}^{m\times d}$ and a vector $b \in \mathbb{R}^m$ such that the primal problem has no solution. Find a solution of the dual problem in your example.
(5 points)