Grading guideline - written examination TMA4180 Optimisation I

Problem 1 (25 points)

(a) 10 points

- **3 p** Correct gradient ∇f and Hessian $\nabla^2 f$.
- **3 p** Identifies $(2, 2)$ as the unique stationary point ($\nabla f = 0$).
- **2 p** Hessian positive definite at $(2, 2)$ / second-order sufficient condition.
- **2 p** Global minimiser argument .

(b) 5 points

- **2 p** Correct conclusion: f is not globally convex.
- **3 p** Valid justification: $\nabla^2 f$ is not positive semidefinite somewhere, e.g. $\det \nabla^2 f < 0$ at a point such as $(0, 1)$.

(c) 10 points

- **2 p** Direction $p = -\nabla f$, with $\nabla f(0, 0) = (-4, 4)$.
- **4 p** Trial $\alpha = 1$: $f(4, -4) = 328$, Armijo condition fails.
- **3 p** Trial $\alpha = 0.25$: $f(1, -1) = 1$, Armijo condition holds.
- **1 p** Final iterate $x_1 = (1, -1)$.

Comment: Claiming global convexity from positive-definiteness at a single point is a substantive error and costs the justification points in (b).

Problem 2 (15 points)

(a) 6 points

- **2 p** LICQ: $\nabla c = (1, 1) \neq 0$.
- **1 p** KKT / stationarity system set up.
- **3 p** All three points $(1, 0)$, $(0, 0)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ with multipliers; spurious and negative-multiplier roots rejected.

(b) 6 points

- **1 p** Lagrangian Hessian $\nabla^2_{xx} L = \nabla^2 f$.
- **2 p** Critical cone at each point.
- **3 p** Verdicts: $(1, 0)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ strict minima; $(0, 0)$ fails the second-order condition on its critical cone.

(c) 3 points

- **3 p** Identifies $(1, 0)$ as the global minimiser (comparison of function values).

Comment: mis-factoring the stationarity equation ($2x_1^2 - 1$ vs $x_1^2 - 1$) loses the third KKT point and is penalised in (a). Declaring $(0, 0)$ a minimiser without the critical-cone test is the most common error in (b).

Problem 3 (15 points)

(a) 4 points

- **4 p** Penalty function $Q(x;\mu) = f(x) + \dfrac{\mu}{2}\,c(x)^2$ with the correct f, c and sign.

(b) 6 points

- **2 p** Sets up $\nabla Q = 0$.
- **4 p** Correct minimiser $x(\mu) = \left(\dfrac{2\mu}{3\mu+2},\ -\dfrac{\mu}{3\mu+2}\right)$.

(c) 5 points

- **2 p** Limit $\mu \to \infty$ gives $\left(\frac{2}{3}, -\frac{1}{3}\right)$.
- **3 p** Verifies against the true constrained minimiser (Lagrange / KKT).

Comment: a wrong sign or a linear penalty term in (a) propagates into (b); deduct once and still give credit in (c) where the limit is recovered by a Lagrange argument.

Problem 4 (15 points)

- **3 p** (a) Derivation of the step length $\alpha_k = -\dfrac{r_k^\mathsf{T} p_k}{p_k^\mathsf{T} A\, p_k}$.
- **3 p** (b) Linear independence of the conjugate directions (premultiply $\sum_i c_i p_i = 0$ by $p_j^\mathsf{T} A$ and use conjugacy).
- **3 p** (c) $a_k = -\dfrac{p_k^\mathsf{T} r_0}{p_k^\mathsf{T} A\, p_k}$ via conjugacy.
- **3 p** (d) $p_k^\mathsf{T} r_k = p_k^\mathsf{T} r_0$.
- **3 p** (e) Convergence in at most n steps (the directions form a basis of $\mathbb{R}^n$).

Comment: the $\frac{1}{2}$-convention for $\varphi(x) = \frac{1}{2}\,x^\mathsf{T} A x - b^\mathsf{T} x$ is accepted. In (b), arguing from orthogonality of residuals rather than conjugacy of directions is incomplete and is penalised.

Problem 5 (15 points)

- **3 p** (a) Choice of B_0 (e.g. the identity) with one valid advantage and one valid disadvantage.
- **5 p** (b) From the secant equation, $v = \delta\,(y_k - B_k s_k)$.
- **4 p** (c) $\delta = \pm\big((y_k - B_k s_k)^\mathsf{T} s_k\big)^{-1/2}$ and the explicit SR1 update.
- **3 p** (d) True/False: (i) False, (ii) True, (iii) True (1 point each).

Comment: in (d), one point per correct answer; if the answer was justified, extra points were given within the exercise to compensate for possible missed posints.

Problem 6 (15 points)

- **2 p** (a) Unboundedness: a feasible sequence (e.g. $(k,\ 4-k)$) whose objective $\to -\infty$.
- **3 p** (b) Standard form: slacks, $c = (-3,-2,0,0)$, $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, $b = (4,6)$.
- **3 p** (c) Dual formulation: max $b^\mathsf{T}\lambda$ subject to $A^\mathsf{T}\lambda \le c$.
- **3 p** (c) Solving the dual: $\lambda^* = (-1,-1)$, optimal value -10; strong duality.
- **2 p** (d) Sketch of the feasible region.
- **2 p** (d) Basic feasible points $(0,0)$, $(3,0)$, $(2,2)$, $(0,4)$ and the optimal value at $(2,2)$.

Comment: a frequent shortfall is setting up the dual but not solving it; the formulation earns the first 3 *points only, and the optimal multipliers and value are needed for the rest.*

Grade boundaries

A : 89–100, B : 77–88, C : 65–76, D : 53–64, E : 41–52, F : 0–40.

Cases at the boundaries are given extra attention.