Problem 1.

Consider the unconstrained optimisation problem

$$\min_{x,y} f(x, y),$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x, y) = x^4 - 2x^2y + 2y^2 + 2x^2 + 4y - 3.$$

a) Find all local and global minimisers of f.
(10 points)

b) Determine whether the function f is convex.
(5 points)

c) Perform one step of Newton's method with backtracking Armijo line search starting from the point $(x, y) = (1, -1)$. Use the parameters $c_1 = 0.1$ (sufficient decrease parameter) and $\rho = 0.5$ (contraction factor).
(10 points)

Problem 2.

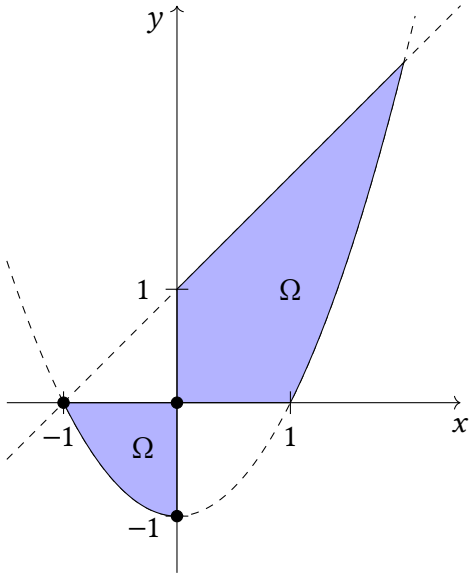
Consider the optimisation problem

$$\min_{x,y} 2x^2 - y^2$$

subject to the constraint $(x, y) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is given by the inequalities

$$xy \geq 0, \qquad y \geq x^2 - 1, \qquad \text{and} \qquad x \geq y - 1.$$

Here is a sketch of the set Ω:



a) Determine for each point in Ω whether the linear independence constraint qualification (LICQ) holds.
(5 points)

b) Determine the tangent cone $T_\Omega(x, y)$ and the cone of linearised feasible directions $\mathcal{F}(x, y)$ to the set Ω in the points $(x, y) = (0, -1)$, $(x, y) = (0, 0)$, and $(x, y) = (-1, 0)$. (The points are marked in the sketch.)
(10 points)

c) Find all KKT points and all local minimisers for this optimisation problem.
(15 points)

Problem 3.

Assume that $f_i\colon \mathbb{R}^d \to \mathbb{R}$, $i = 1, \dots, n$, are continuously differentiable, convex functions. Define the function $f\colon \mathbb{R}^d \to \mathbb{R}$,

$$f(x) := \max\{f_1(x), \dots, f_n(x)\}.$$

We consider the problem

$$\min_{x\in\mathbb{R}} f(x). \tag{1}$$

a) Show that the problem (1) is equivalent to the problem

$$\min_{(x,t)\in\mathbb{R}^d\times\mathbb{R}} t \qquad \text{s.t. } t \geq f_i(x) \qquad \text{for all } i = 1, \dots, n. \tag{2}$$

in the sense that x^* solves (1), if and only if (x^*, t^*) solves (2) and $t^* = f(x^*)$.

(5 points)

b) Use part a) to show that $x^* \in \mathbb{R}^d$ solves (1), if and only if there exist $\lambda_i^* \geq 0$, $i \in A(x^*)$, such that

$$\sum_{i\in A(x^*)} \lambda_i^* \nabla f_i(x^*) = 0 \qquad \text{and} \qquad \sum_{i\in A(x^*)} \lambda_i^* = 1,$$

where $A(x^*) := \{i : f_i(x^*) = f(x^*)\}$.

(10 points)

Problem 4.

Let $d \gg 1$ and let $b \in \mathbb{R}^d$. Define the function $f\colon \mathbb{R}^d \to \mathbb{R}$,

$$f(x) := \sum_{i=1}^{d} (x_i - b_i)^2 + \sum_{i=1}^{d-1} \sqrt{1 + (x_{i+1} - x_i)^2}.$$

Propose a numerical method for the solution of the problem

$$\min_{x\in\mathbb{R}^d} f(x).$$

Justify your proposal!

(10 points)

Problem 5.

Let $c \in \mathbb{R}^d$, $A \in \mathbb{R}^{m \times d}$, and $b \in \mathbb{R}^m$, and consider the constrained optimisation problem

$$\min_x \frac{1}{2}\|x\|_2^2 + \langle c, x \rangle \qquad \text{s.t. } Ax \geq b. \tag{3}$$

Find the Lagrangian dual of the problem (3).

(10 points)

Problem 6.

Consider the Fletcher–Reeves non-linear CG method for the solution of the optimisation problem $\min_x f(x)$. Assume that the step length $\alpha_k > 0$ in each step is chosen according to an *exact* line search. Show that p_k is in each step a descent direction provided that $\nabla f(x_k) \neq 0$.

(10 points)