Problem 1.

Consider the unconstrained optimisation problem

$$\min_{x,y} f(x, y),$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x, y) = x^3 + 7x^2 - 2xy + y^2 + 3x + 6y.$$

a) Find all local and global minimisers of f.
(10 points)

b) Determine whether the function f is convex.
(5 points)

c) Perform one step of Newton's method with backtracking Armijo line search starting from the point $(x, y) = (0, 0)$. Use the parameters $c_1 = 0.1$ (sufficient decrease parameter) and $\rho = 0.5$ (contraction factor).
(10 points)

Problem 2.

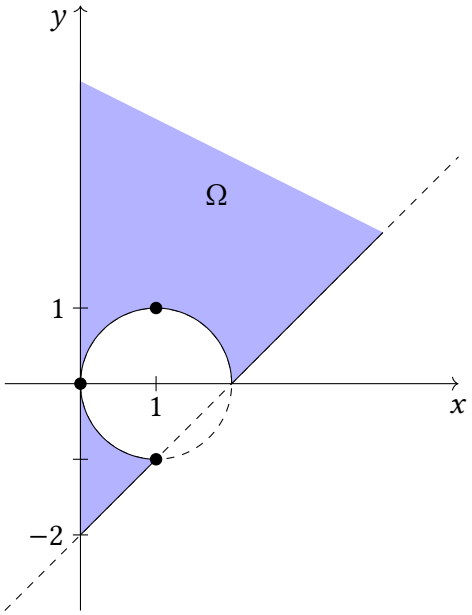
Consider the optimisation problem

$$\min_{x,y} (x-1)^2 + (y+2)^2$$

subject to the constraint $(x, y) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is given by the inequalities

$$x \geq 0, \qquad y \geq x - 2, \qquad \text{and} \qquad (x-1)^2 + y^2 \geq 1.$$

Here is a sketch of the set Ω (it continues infinitely to the top right):



a) Determine for each point in Ω whether the LICQ holds.
(5 points)

b) Determine the tangent cone $T_\Omega(x, y)$ and the cone of linearised feasible directions $\mathcal{F}(x, y)$ to the set Ω in the points $(x, y) = (1, -1)$, $(x, y) = (1, 1)$, and $(x, y) = (0, 0)$. (The points are marked in the sketch.)
(10 points)

c) Find all KKT points and all local minimisers for this optimisation problem. Use the second order necessary / sufficient conditions whenever applicable.
(15 points)

Problem 3.

Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable and bounded below. We consider the gradient descent method

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

for the minimisation of f, where the step lengths $\alpha_k > 0$ are chosen such that the Armijo condition with fixed parameter $c_1 > 0$ is satisfied in each step. Assume moreover that there exists $\varepsilon > 0$ such that

$$\alpha_k \geq \varepsilon > 0 \qquad \text{for all } k.$$

Show that

$$\lim_{k\to\infty} \nabla f(x_k) = 0.$$

(10 points)

Problem 4.

Define the functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$,

$$f_1(x, y) = (x-1)^2 + (y-2)^2, \qquad f_2(x, y) = x^2 + (y+2)^2.$$

Find all Pareto-optimal solutions of the multi-criteria optimisation problem

$$\min_{(x,y)\in\mathbb{R}^2} \big(f_1(x, y), f_2(x, y)\big).$$

(10 points)

Problem 5.

Assume that $f\colon \mathbb{R}^d \to \mathbb{R}$ is convex, continuous, and bounded below, and that $A \in \mathbb{R}^{m\times d}$ with $m < d$ has full rank. We define the function $g\colon \mathbb{R}^m \to \mathbb{R}$,

$$g(y) := \inf_{\substack{x\in\mathbb{R}^d \\ Ax=y}} f(x).$$

Show that g is convex!

(10 points)

Problem 6.

Let $a \in \mathbb{R}^d$ be fixed and consider the constrained optimisation problem

$$\min_{x\in\mathbb{R}^d} \sum_{i=1}^{d} \exp(x_i) \qquad \text{s.t.} \begin{cases} x \le 0, \\ \langle a, x\rangle \le 1. \end{cases} \tag{1}$$

a) Formulate the KKT conditions for (1). Moreover, show that (1) has no solution if there exists any $1 \le i \le d$ such that $a_i \ge 0$.
(5 points)

b) Compute the Lagrangian dual of the optimisation problem (1). Does the dual problem have a solution?
(10 points)