Exercise 9

March 12, 2024

Problem 1.

Let

$$f(x) = x_1^4 + 2x_2^4 + x_1x_2 + x_1 - x_2 + 2.$$

Starting at the point $x_0 = (0, 0)$ compute explicitly one step for the trust region method with the model function

$$m(p) = f(x_0) + \langle g, p\rangle + \frac{1}{2}\langle p, Bp\rangle$$

where $g = \nabla f(x_0)$, $B = \nabla^2 f(x_0)$, and the trust region radius is $\Delta = 1$.

Problem 2.

Let

$$f(x) = \frac{1}{2}x_1^2 + x_2^2,$$

let $x_0 = (1, 1)$, and define the model function $m(p) = f(x_0)+\langle g, p\rangle+\frac{1}{2}\langle p, Bp\rangle$ with $g = \nabla f(x_0)$ and $B = \nabla^2 f(x_0)$.

a) Compute explicitly the next step p in the trust region method using values of $\Delta = 2$ and $\Delta = 5/6$.

b) Compute for all $\Delta > 0$ the next step in the dogleg method.

Problem 3. (Problem 4.1 in N&W)

Let

$$f(x) = 10(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

At $x = (0, -1)$ draw the contour lines of the quadratic model

$$\min_p m(p) = f(x) + \langle \nabla f(x), p\rangle + \frac{1}{2}\langle p, Bp\rangle \qquad \text{subject to } \|p\| \le \Delta, \quad (1)$$

assuming that B is the Hessian of f. Draw the family of solutions of (1) as the trust region radius varies from $\Delta = 0$ to $\Delta = 2$. Repeat this at $x = (0, 0.5)$.

Problem 4. (Problem 4.5 in N&W)

Let $\phi(\tau) = m_k(\tau p_k^s)$, where $p_k^s = -\frac{\Delta_k}{\|g_k\|} g_k$ and $m_k(p) = f_k + g_k^T p + \frac{1}{2} p^T B p$ with $B \in \mathbb{R}^{d \times d}$ symmetric. Show that the minimiser of $\phi(\tau)$ subject to $\|\tau p_k^s\| \leq \Delta_k$ and $\tau \geq 0$ is given as

$$\begin{cases} 1, & \text{if } \langle g_k, B_k g_k \rangle \leq 0, \\ \min\left\{ \dfrac{\|g_k\|^3}{\Delta_k \langle g_k, B_k g_k \rangle}, 1 \right\}, & \text{otherwise.} \end{cases} \tag{2}$$