Exercise 7

February 27, 2024

Problem 1.

Consider the constrained optimisation problem

$$\min_{(x,y)}(xy) \qquad \text{such that} \qquad \begin{cases} y \geq x, \\ y^4 \leq x^3. \end{cases}$$

(Note that the constraint set is the same as in Exercise 6, Problem 1.)

a) Find all KKT points and local minima for this optimisation problem.

b) Compute the critical cone at $(0,0)$ as defined in the lecture and Nocedal & Wright, and show that there exist directions p contained in the critical cone for which $p^T \nabla^2 \mathcal{L}((0,0), \lambda^*)p < 0$.

c) Show that $p^T \nabla^2 \mathcal{L}((0,0), \lambda^*)p \geq 0$ for all vectors p contained in the tangent cone to the feasible set at $(0,0)$.

Problem 2.

Consider the constrained optimisation problem

$$\min_{(x,y)} \frac{1}{2}(x^2 + y^2) \qquad \text{subject to } xy = 1.$$

a) Find (by whatever means) the solutions of this problem. In addition, find the values of the corresponding Lagrange multipliers.

b) Formulate the unconstrained optimisation problem that results from the application of the quadratic penalty method with parameter $\mu > 0$. Solve these problems for all possible parameters μ and verify that the solutions converge to the solutions of the constrained optimisation problem as $\mu \to \infty$.

c) Formulate the augmented Lagrangian for this constrained optimisation problem and find (for all possible parameters $\lambda \in \mathbb{R}$ and $\mu > 0$) the global solutions of this (unconstrained) optimisation problem. For which parameters does one recover the solution of the original constrained problem?

Problem 3.

Consider the constrained optimisation problem

$$\min_{x,y}(x+y) \qquad \text{such that } x^2+y^2 \leq 1.$$

Formulate a logarithmic barrier method for the solution of this constrained optimisation problem and compute its solution for each parameter $\mu > 0$ in the barrier functional.

Problem 4.

Assume that $A \in \mathbb{R}^{m\times n}$ with $m < n$ is a matrix of full rank and that $b \in \mathbb{R}^m \setminus \{0\}$. Consider the optimisation problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad Ax = b. \tag{1}$$

a) Formulate the KKT-conditions for this problem and show that the unique solution is given by

$$x^* = A^T(AA^T)^{-1}b.$$

b) Formulate the quadratic penalty method for this constrained optimisation problem, and show that the unique minimiser with parameter $\mu > 0$ is given by

$$x_\mu := A^T\Big(\frac{1}{\mu}\,\mathrm{Id} + AA^T\Big)^{-1}b$$

with $\mathrm{Id} \in \mathbb{R}^{m\times m}$ denoting the identity matrix.

c) Now consider the optimisation problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad \frac{1}{2}\|Ax-b\|^2 \leq \varepsilon$$

for some $\varepsilon > 0$, and denote its solution by $\hat{x}_\varepsilon$. Show that either $\frac{1}{2}\|b\|^2 \leq \varepsilon$ (in which case $\hat{x}_\varepsilon = 0$), or there exists $\mu > 0$ such that $\hat{x}_\varepsilon = x_\mu$.