Exercise 6

February 20, 2024

Problem 1.

Sketch the region

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : y \geq x \text{ and } y^4 \leq x^3\right\}$$

and compute the tangent cone and the set of linearised feasible directions for each point in Ω. For which points in Ω is the LICQ satisfied?

Problem 2.

Assume that one wants to solve the optimisation problem

$$\max_x f(x) \qquad \text{such that} \qquad \begin{cases} c_i(x) = 0 & \text{for all } i \in \mathcal{E}, \\ c_i(x) \geq 0 & \text{for all } i \in \mathcal{I}. \end{cases}$$

How can we modify the KKT conditions such that one obtains (first order) necessary conditions for this maximisation problem?

Problem 3.

Consider the constrained optimization problem

$$\min_{(x,y)}(x^2 + y^2) \qquad \text{such that} \qquad \begin{cases} x + y \geq 1, \\ y \leq 2, \\ y^2 \geq x. \end{cases}$$

a) Formulate the KKT-conditions for this optimisation problem.

b) Find all KKT points.

c) Find all local and global minima.

(Part b) can be very tedious. One strategy is to consider all possible active sets and determine for each active set whether KKT-points exist. It can also be extremely helpful to sketch the feasible set and the function in order to identify the different possibilities for active sets.)

Problem 4.

Consider the constrained optimization problem

$$\min_{(x,y)} x \qquad \text{such that} \qquad \begin{cases} y \geq x^4, \\ y \leq x^3. \end{cases}$$

Find all KKT points and local minima for this optimisation problem.

Problem 5.

Consider the constrained optimization problem

$$\min_{(x,y)} \left(-x^2 - (y-1)^2\right) \qquad \text{such that} \qquad \begin{cases} y \geq Cx^2, \\ y \leq 2, \end{cases}$$

where $C > 0$ is some positive parameter.

a) Show that the point $(0, 0)$ is a KKT point for all parameters $C > 0$ and that the LICQ is satisfied at $(0, 0)$.

b) Formulate the second order necessary and sufficient optimality conditions for the point $(0, 0)$. For which parameters C are these conditions satisfied? For which parameters C is the point $(0, 0)$ a local minimum?