Exercise 1

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Problem 1.

Assume that $f, g\colon \mathbb{R}^d \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ are extended real valued functions and denote by $L_f(\alpha)$, $L_g(\alpha)$ for $\alpha \in \bar{\mathbb{R}}$ the level sets of f and g, respectively.

a) Show that $L_f(\alpha) \subset L_f(\beta)$ whenever $-\infty \leq \alpha < \beta \leq +\infty$.

b) Assume that $f(x) \geq g(x)$ for all $x \in \mathbb{R}^d$. Show that $L_f(\alpha) \subset L_g(\alpha)$ for all $\alpha \in \bar{\mathbb{R}}$.

c) Define $h\colon \mathbb{R}^d \to \bar{\mathbb{R}}$ by $h(x) := \max\{f(x), g(x)\}$ for $x \in \mathbb{R}^d$. Show that $L_h(\alpha) = L_f(\alpha) \cap L_g(\alpha)$ for all $\alpha \in \bar{\mathbb{R}}$.

Solution.

a) Let $x \in L_f(\alpha)$. Then $f(x) \leq \alpha < \beta$. It follows that $f(x) \leq \beta$, implying that $x \in L_f(\beta)$.

b) Let $x \in L_f(\alpha)$. Then $f(x) \leq \alpha$, which, together with the inequality $g(x) \leq f(x)$, implies $g(x) \leq \alpha$. This gives $x \in L_g(\alpha)$.

c) First, we show that $L_h(\alpha) \subset L_f(\alpha) \cap L_g(\alpha)$.
Let $x \in L_h(\alpha)$. Then $h(x) = \max\{f(x), g(x)\} \leq \alpha$. In order for the maximum of $f(x)$ and $g(x)$ to be less than α, we need both of them to be less than α. Thus we have both $x \in L_f(\alpha)$, and $x \in L_g(\alpha)$, meaning $x \in L_f(\alpha) \cap L_g(\alpha)$.

Now, we show that $L_f(\alpha) \cap L_g(\alpha) \subset L_h(\alpha)$.
Let $x \in L_f(\alpha) \cap L_g(\alpha)$, then we have both $x \in L_f(\alpha)$, and $x \in L_g(\alpha)$. This implies both $f(x) \leq \alpha$ and $g(x) \leq \alpha$, meaning $\max\{f(x), g(x)\} \leq \alpha$, giving us $h(x) \leq \alpha$. Finally, we get $x \in L_h(\alpha)$.

Now we have both $L_f(\alpha) \cap L_g(\alpha) \subset L_h(\alpha)$, and $L_h(\alpha) \subset L_f(\alpha) \cap L_g(\alpha)$. These together imply $L_h(\alpha) = L_f(\alpha) \cap L_g(\alpha)$.

Bonus challenge: if we instead define $h(x)$ as $h(x) = \min\{f(x), g(x)\}$, can you find a similar relationship between the level sets of $f(x)$, $g(x)$ and $h(x)$?

Problem 2.

For each of the following functions, check whether it is lower semi-continuous and/or coercive, and whether it admits a global minimum:

a) $\ell\colon \mathbb{R} \to \mathbb{R}$ defined as $\ell(x) = 5x^{10} + 8x^7 - 9x^2 + x + c$, where $c \in \mathbb{R}$ is a constant.

b) $m\colon \mathbb{R} \to \mathbb{R}$ defined as $m(x) = e^x - \frac{1}{1+x^2}$.

c) $p\colon \mathbb{R} \to \mathbb{R}$ defined as $p(x) = x^4 - 20x^3 + \sup_{k\in\mathbb{N}} \sin(k^2 x)$.

d) $q\colon \mathbb{R}^2 \to \mathbb{R}$ defined as $q(x) = x_1^2(1 + x_2^3) + x_1^2$.

Solution.

a) The function ℓ is a polynomial and hence continuous. Therefore, ℓ is a lower semi-continuous function. Moreover, it is coercive too because of the term x^{10}, which dominates all other terms for large $|x|$. Consequently, a global minimiser exists.

b) The given function m is continuous and thus also semi-continuous. However, it is not coercive because $\lim_{x\to-\infty} g(x) = 0 \neq +\infty$. On the other hand, the closed set $\Omega = \{x \in \mathbb{R} : g(x) \leq g(-1)\}$ is bounded (and therefore compact), since $g(-1) < 0 = \lim_{x\to-\infty} g(x)$ and $g(-1) < 0 < +\infty = \lim_{x\to+\infty} g(x)$. Therefore, the global minimum of g is attained over Ω.

c) Since $x^4 - 20x^3$ is a polynomial, it is continuous. Further $\sin(k^2 x)$ is continuous for all k, implying it is lower semi-continuous as well. Thus $\sup_{k\in\mathbb{N}} \sin(k^2 x)$ is semi-continuous as a supremum of continuous functions.
Now $p(x)$ is the sum of lower semi-continuous function and hence lower semi continuous. The function $p(x)$ is coercive, because of the term x^4 that dominates all other terms for large $|x|$. Consequently, a global minimizer exists.

d) The given function $q(x)$ is a polynomial and hence continuous, which implies its lower semi-continuity. However, it is not bounded below because for any fixed $x_1 \neq 0$ and $x_2 \to -\infty$ we get $q(x) \to -\infty$. Therefore, it does not attain global minimizers on $\mathbb{R}^2$.

Problem 3.

Find the gradient, Hessian, and local minimizers of the objective function f of the optimization problem $\min_{(x,y)\in\mathbb{R}^2} f(x, y)$, where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined as:

a) $f(x,y) = \frac{x^2}{2} + x\cos y$.

b) $f(x,y) = 2x^2 - 4xy + y^4 + 5y^2 - 10y$.

Solution.

a) We compute the gradient and the Hessian of f as:

$$\nabla f(x,y) = \begin{bmatrix} x + \cos y \\ -x\sin y \end{bmatrix}, \quad \text{and} \quad H_f(x,y) = \begin{bmatrix} 1 & -\sin y \\ -\sin y & -x\cos y \end{bmatrix}.$$

The first-order necessary optimality condition gives $\nabla f(x,y) = 0$. Now, we have two sets of critical points:

1. $x = 0$ and $y = (2n+1)\frac{\pi}{2}$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. In this case the Hessian matrix is

$$H_f(x,y) = \begin{bmatrix} 1 & (-1)^{n+1} \\ (-1)^{n+1} & 0 \end{bmatrix}.$$

Since the determinant of this matrix is always negative, it follows that it is indefinite, and thus these points are not local solutions.

2. $x = (-1)^{n+1}$ and $y = n\pi$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. In this case the Hessian matrix is

$$H_f(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is clearly positive definite and hence the above mentioned critical points are strict local minimisers.

b) We compute the gradient and the Hessian of f as:

$$\nabla f(x,y) = \begin{bmatrix} 4x - 4y \\ -4x + 4y^3 + 10y - 10 \end{bmatrix}$$

and

$$H_f(x,y) = \begin{bmatrix} 4 & -4 \\ -4 & 12y^2 + 10 \end{bmatrix}.$$

The first-order necessary optimality condition gives $\nabla f(x,y) = 0$. Now, we have the critical point $x = (1,1)$ where the Hessian matrix is

$$\nabla^2 f(x,y) = \begin{bmatrix} 4 & -4 \\ -4 & 22 \end{bmatrix}.$$

Since the diagonal entries of this matrix are positive and so is the determinant, it is positive definite. Therefore, $(1,1)$ is a strict local minimiser.

Problem 4.

Compute the gradient, Hessian and local minimisers of the Rosenbrock function $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.

Solution.

The gradient and Hessian of f are

$$\nabla f(x_1, x_2) = \begin{bmatrix} -400x_1(x_2 - x_1^2) - 2(1 - x_1) \\ 200(x_2 - x_1^2) \end{bmatrix}$$

and

$$H_f(x_1, x_2) = \begin{bmatrix} -400x_2 + 1200x_1^2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}.$$

The first-order necessary optimality condition gives $\nabla f(x, y) = 0$. Now, we have the solution $x = (1, 1)$, where we have the Hessian

$$H_f(x, y) = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}.$$

Since the diagonal entries are positive and so is the determinant, this matrix is positive definite. Thus $(1, 1)$ is a strict local minimiser.

Problem 5.

For a matrix $A \in \mathbb{R}^{d\times d}$, we denote by

$$\|A\|_F := \left(\sum_{i,j=1}^{d} a_{ij}^2 \right)^{\frac{1}{2}}$$

its *Frobenius norm*. Show that the optimization problem

$$\min_{\substack{A\in\mathbb{R}^{d\times d} \\ \det A>0}} \left(\|A\|_F + \frac{1}{\det A} \right)$$

admits a global minimum.

Solution.

First, we discuss the continuity of the objective function on the feasible set: The objective function $f(A) = \|A\|_F + \frac{1}{\det A}$ is continuous because the Frobenius norm is continuous and the determinant is also continuous (because it is polynomial of matrix elements). Thus $f(A)$ is continuous over the feasible set $\Omega = \{A \in \mathbb{R}^{d\times d} : \det A > 0\}$, which is neither closed nor bounded, though.

Now we aim to construct a closed and bounded set, say $S \subset \Omega$ in which the objective function f is continuous, and such that if a global minimum exists, it must be contained in S.
For that, we note that the identity matrix $I \in \mathbb{R}^{d\times d}$ satisfies $I \in \Omega$ with $f(I) = d^{\frac{1}{2}} + 1$. Therefore, if global minimum exists, then it should be in the set $\{A \in \Omega : f(A) \leq f(I)\} = \{A \in \Omega : f(A) \leq d^{\frac{1}{2}} + 1\} \subset \{A \in \mathbb{R}^{d\times d} : \|A\|_F \leq d^{\frac{1}{2}} + 1\} \cap \{A \in \Omega : \det A \geq (d^{\frac{1}{2}} + 1)^{-1}\}$. We now define

$$S = \{A \in \mathbb{R}^{d\times d} : \|A\|_F \leq d^{\frac{1}{2}} + 1\} \cap \{A \in \Omega : \det\, A \geq (d^{\frac{1}{2}} + 1)^{-1}\}$$

which is bounded (the first set in the intersection is bounded) and closed (both intersected sets are closed, since the functions defining the inequality constraints are continuous). Therefore, the given optimisation problem attains a global minimum in S and hence in Ω too.