# Exercise #12

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**Problem 1.**

Find (and simplify, if possible) the dual of the linear programme

$$\min c^T x \qquad \text{subject to } Ax \geq b,\ x \geq 0.$$

**Solution.**

The Lagrangian of the problem is

$$\begin{aligned} \mathcal{L}(x, \lambda, s) &= c^T x - \lambda^T (Ax - b) - s^T x \\ &= b^T \lambda + (c - A^T \lambda - s)^T x, \end{aligned}$$

and the dual problem is defined as

$$\max_{\lambda \geq 0, s \geq 0} \min_x \mathcal{L}(x, \lambda, s). \tag{1}$$

Since

$$\min_x \mathcal{L}(x, \lambda, s) = \begin{cases} -\infty & \text{if } A^T \lambda + s \neq c; \\ b^T \lambda & \text{if } A^T \lambda + s = c, \end{cases}$$

we see that (1) is equivalent to the problem

$$\max_{\lambda \geq 0, s \geq 0} b^T \lambda \qquad \text{subject to} \qquad A^T \lambda + s = c.$$

Interpreting $s$ as a slack variable, we can further simplify the dual problem to

$$\max_\lambda b^T \lambda \qquad \text{subject to} \qquad A^T \lambda \leq c,\ \lambda \geq 0.$$

**Problem 2.**

Find the dual of the linear optimisation problem

$$5x_1 + 3x_2 + 4x_3 \to \min \qquad \text{subject to } \begin{cases} x_1 + x_2 + x_3 = 1, \\ \qquad\qquad x_i \geq 0, \quad i = 1, 2, 3, \end{cases}$$

and compute its (i.e., the *dual's*) solution. Use the dual's solution in order to find a solution of the original problem.

**Solution.**

The linear optimisation problem may be written as

$$\min_x c^T x \qquad \text{subject to} \qquad Ax = b,\ x \ge 0,$$

and its dual is, similarly as in the previous exercise,

$$\max_\lambda b^T \lambda \qquad \text{subject to} \qquad A^T \lambda \le c.$$

Note that the constraint $\lambda \ge 0$ is not present in this case, because we have an equality constraint $Ax = b$, and not an inequality constraint $Ax \ge b$. With

$$c = (5, 3, 4), \quad A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad b = 1,$$

the dual for this specific problem becomes

$$\max \lambda \qquad \text{subject to} \qquad \lambda \le 5,\ \lambda \le 3,\ \lambda \le 4,$$

with obvious solution $\lambda^* = 3$.

In order to compute the primal solution, we now note that $x^*$ will act as a Lagrange multiplier for the dual problem. Since only the second constraint ($\lambda \le 3$) is active in the dual solution, it follows that $x_1^* = 0$ and $x_3^* = 0$. Moreover, $x^*$ satisfies the constraint $x_1^* + x_2^* + x_3^* = 1$, which now implies that $x_2^* = 1$. The solution of the primal problem thus is $x^* = [0\ 1\ 0]$.

**Problem 3.**

Assume that $A \in \mathbb{R}^{m\times n}$ with $m < n$ is a matrix of full rank and that $b \in \mathbb{R}^m \setminus \{0\}$. Consider the optimization problem

$$\frac{1}{2}\|x\|^2 \longrightarrow \min \qquad \text{subject to} \qquad Ax = b. \tag{2}$$

a) Compute the solution of (2).

b) Derive an explicit formula for the dual problem to (2).

c) Show that $\lambda^* \in \mathbb{R}^m$ solves the dual problem, if and only if

$$AA^T\lambda^* = b.$$

d) Verify directly that in this situation

$$\min_{x\in\mathbb{R}^n} \max_{\lambda\in\mathbb{R}^m} \mathcal{L}(x,\lambda) = \max_{\lambda\in\mathbb{R}^m} \min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda).$$

**Solution.**

a) We have a strictly convex optimisation problem with linear constraints. Thus the solution $x^*$ is uniquely characterised by the KKT conditions. The Lagrangian of the problem is defined as

$$\mathcal{L}(x,\lambda) = \frac{1}{2}\|x\|^2 - \lambda^T(Ax - b),$$

where $\lambda \in \mathbb{R}^m$. The KKT conditions thus read

$$x^* - A^T\lambda^* = 0 \qquad \text{and} \qquad Ax^* = b.$$

Inserting the first term into the second and solving for $\lambda^*$, we obtain that

$$\lambda^* = (AA^T)^{-1}b.$$

By inserting this solution into the first term, we thus see that

$$x^* = A^T(AA^T)^{-1}b.$$

b) We start by minimising the Lagrangian with respect to $x$. Calculating

$$\nabla_x \mathcal{L}(x,\lambda) = x - A^T\lambda \qquad \text{and} \qquad \nabla_x^2 \mathcal{L}(x,\lambda) = \mathrm{Id}_{n\times n}$$

shows that $\mathcal{L}(\cdot,\lambda)$ is positive definite and has a unique minimiser $x^* = A^T\lambda$. Thus the dual problem is

$$\max_{\lambda\in\mathbb{R}^m} q(\lambda),$$

where

$$q(\lambda) = \min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda) = \mathcal{L}(x^*,\lambda) = \frac{1}{2}\|A^T\lambda\|^2 - \lambda^T(AA^T\lambda - b) = b^T\lambda - \frac{1}{2}\|A^T\lambda\|^2.$$

c) Observe first that

$$\nabla q(\lambda) = b - (A^T)^T A^T\lambda = b - AA^T\lambda \qquad \text{and} \qquad \nabla^2 q(\lambda) = -AA^T.$$

Since $A$ has full rank $m$, the matrix $AA^T$ is positive definite. Hence, $q$ is negative definite, and there exists a unique maximiser $\lambda^*$ of the dual problem satisfying $\nabla q(\lambda^*) = 0$, that is, $AA^T\lambda^* = b$. (A clarification: "there exists" is the "if" part of the question, while "unique" is the "only if" part.)

d) The primal problem

$$\max_{\lambda\in\mathbb{R}^m}\min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda)$$

is equivalent to the original optimisation problem, which has the solution $x^* = A^T(AA^T)^{-1}b$. As such, the optimal value of the primal problem becomes

$$\tfrac{1}{2}\|x^*\|^2 = \frac{1}{2}b^T(AA^T)^{-T}A\,A^T(AA^T)^{-1}b = \tfrac{1}{2}b^T(AA^T)^{-1}b,$$

because $(AA^T)^{-T} = \big((AA^T)^T\big)^{-1} = (AA^T)^{-1}$. From b) we have $\lambda^* = (AA^T)^{-1}b$, so the optimal value of the dual problem also equals

$$q(\lambda^*) = \tfrac{1}{2}b^T(AA^T)^{-1}b,$$

after performing similar cancellations.

**Problem 4.**

Consider the linear programme

$$c^Tx \longrightarrow \min \qquad\qquad \text{subject to } Ax = b \text{ and } x \geq 0,$$

where $c \in \mathbb{R}^d$, $A \in \mathbb{R}^{m\times d}$, and $b \in \mathbb{R}^m$. We can approximate this problem using a logarithmic barrier function, which results in the new problem

$$f(x) - \beta\sum_i \ln(x_i) \qquad\qquad \text{subject to } Ax = b. \tag{3}$$

Here we set $-\ln(t) = +\infty$ for $t \leq 0$.

Compute the Lagrangian dual of (3).

**Solution.**

The Lagrangian of the problem is

$$\mathcal{L}(x,\lambda) = c^T x - \beta \sum_i \ln(x_i) - \langle \lambda, Ax - b\rangle.$$

The dual objective functional is thus defined as

$$q(\lambda) = \inf_{x>0}\Big(c^T x - \beta \sum_i \ln(x_i) - \langle \lambda, Ax - b\rangle\Big) = \langle \lambda, b\rangle + \inf_{x>0}\Big(c^T x - \beta \sum_i \ln(x_i) - \langle A^T\lambda, x\rangle\Big).$$

Here we only have to maximise over positive values of $x_i$, since the Lagrangian is $+\infty$ else. We can further simplify this expression to

$$q(\lambda) = \langle \lambda, b\rangle + \sum_i \Big(\inf_{x_i>0}\big(c_i x_i - (A^T\lambda)_i x_i - \beta \ln(x_i)\big)\Big).$$

Concerning the inner optimisation problem

$$\inf_{x_i>0}\big((c_i - (A^T\lambda)_i)x_i - \beta\ln(x_i)\big),$$

we see that this is unbounded below if $c_i - (A^T\lambda)_i \le 0$. On the other hand, for $c_i - (A^T\lambda)_i > 0$, it has a unique solution $x_i^*$ given by

$$c_i - \frac{\beta_i}{x_i^*} - (A^T\lambda)_i = 0,$$

or

$$x_i^* = \frac{\beta}{c_i - (A^T\lambda)_i}.$$

Thus we obtain that

$$\inf_{x_i>0}\big((c_i - (A^T\lambda)_i)x_i - \beta\ln(x_i)\big) = \begin{cases} \beta - \beta\ln\Big(\dfrac{\beta}{c_i - (A^T\lambda)_i}\Big) & \text{if } (A^T\lambda)_i < c_i, \\ -\infty & \text{if } (A^T\lambda)_i \ge c_i, \end{cases}$$

which can be further rewritten as

$$\inf_{x_i>0}\big((c_i - (A^T\lambda)_i)x_i - \beta\ln(x_i)\big) = \begin{cases} \beta - \beta\ln(\beta) + \beta\ln\big(c_i - (A^T\lambda)_i\big) & \text{if } (A^T\lambda)_i < c_i, \\ -\infty & \text{if } (A^T\lambda)_i \ge c_i. \end{cases}$$

In total, using again the convention that $\ln(t) = -\infty$ for $t \le 0$, we thus obtain the dual objective function

$$q(\lambda) = \langle \lambda, b\rangle + d\beta(1 - \ln(\beta)) + \beta \sum_i \ln\big(c_i - (A^T\lambda)_i\big).$$

The dual problem is therefore the optimisation problem

$$\max_\lambda \langle \lambda, b\rangle + \beta \sum_i \ln\big(c_i - (A^T\lambda)_i\big) \qquad \text{subject to } A^T\lambda < c.$$

Here we have ignored the constant term $d\beta(1 - \ln(\beta))$, which is irrelevant for the solution.