Exercise #11

April 18, 2023

Problem 1.

Decide for each of the following binary relations whether it is a partial order, a total order, or no order relation at all.

a) The relation $\preceq$ on $\mathbb{C}$ given by $x \preceq y$ if $\Re x \leq \Re y$ (here $\Re x$, $\Re y$ denote the real part of x and y, respectively).

b) The relation $\preceq$ on $\mathbb{R}^n$, $n \geq 1$, given by $x \preceq y$ if $\|x\| \leq \|y\|$.

c) The relation $\preceq$ on $\mathbb{R}^n$, $n \geq 1$, given by $x \preceq y$ if $x_1 \leq y_1$ and $x_i = y_i$ for $2 \leq i \leq n$.

d) The relation $\preceq$ on the set of cubic polynomials given by $p \preceq q$ if the largest roots x_p, x_q of p and q, respectively, satisfy $x_p \leq x_q$.

Solution.

a) Let $x = a_1 + ib_1$ and $y = a_2 + ib_2$.
Reflexivity: Obviously, $a_1 \leq a_1$, i.e., $\Re x \leq \Re x$, which implies $x \preceq x$.
Anti-symmetry: Obviously, $a_1 \leq a_2$ and $a_2 \leq a_1$ implies $a_1 = a_2$, i.e., $\Re x \leq \Re y$ and $\Re y \leq \Re x$ implies $\Re x = \Re y$. However, we cannot write
$$x \preceq y \text{ and } y \preceq x \Rightarrow x = y.$$
For example, we have $x = 1 + i$ and $y = 1 + 2i$. Obviously, $\Re x = \Re y$ and $x \preceq y$ but $x \neq y$. Therefore, the given relation $\preceq$ is not anti-symmetric and hence it is not a partial order relation. Moreover, it is not a total order relation either.

b) **Reflexivity:** Obviously, $\|x\| \leq \|x\|$, which implies $x \preceq x$.
Anti-symmetry: Obviously $\|x\| \leq \|y\|$ and $\|y\| \leq \|x\|$ implies $\|x\| = \|y\|$. However, we cannot say
$$x \preceq y \text{ and } y \preceq x \Rightarrow x = y.$$
For example, let $x = (1, -1)$ and $y = (1, 1)$. Then $\|x\| = \|y\| = \sqrt{2}$ and $x \preceq y$ and $y \preceq x$, but $x \neq y$. Therefore, the given relation $\preceq$ is not anti-symmetric and hence it is not a partial order relation. Moreover, it is not a total order relation either.

c) **Reflexivity:** Obviously, $x_1 \leq x_1$ and $x_i \leq x_i$ for $2 \leq i \leq n$, which implies $x \preceq x$.
Anti-symmetry: Obviously, $x_1 \leq y_1$, $x_i = y_i$ for $2 \leq i \leq n$ and $y_1 \leq x_1$, $y_i = x_i$ for $2 \leq i \leq n$ imply $x_1 = y_1$ and $x_i = y_i$ for $2 \leq i \leq n$. Therefore, $x \preceq y$ and $y \preceq x$ yield $x = y$.
Transitivity: Obviously, $x_1 \leq y_1$, $x_i = y_i$ for $2 \leq i \leq n$ and $y_1 \leq z_1$, $y_i = z_i$ for $2 \leq i \leq n$ imply $x_1 \leq z_1$ and $x_i = z_i$ for $2 \leq i \leq n$. Therefore, $x \preceq y$ and $y \preceq z$ yield $x \preceq z$.
Thus, the given relation $\preceq$ is a partial order.
Total order: The given relation $\preceq$ is not a total order. Take for example $x = (x_1, x_2) = (0, 0)$ and $y = (y_1, y_2) = (0, 1)$. Since $x_2 \neq y_2$, neither of the relations $x \preceq y$ or $y \preceq x$ hold.

d) **Reflexivity:** Obviously, $x_p \leq x_p$ which implies $x \preceq x$.

Anti-symmetry: Obviously, $x_p \leq x_q$ and $x_q \leq x_p$ yield $x_p = x_q$. However, we cannot say

$$x \preceq y \text{ and } y \preceq x \Rightarrow x = y.$$

For example, let $p(x) = x^2(x-1)$ and $q(x) = (x-1)^3$. Then the largest roots of p and q are $x_p = x_q = 1$. Thus $p \preceq q$ and $q \preceq p$, but, obviously, $p \neq q$. Therefore, the given relation $\preceq$ is not anti-symmetric and hence it is not a partial order relation. Moreover, it is not a total order relation either.

Problem 2.

On the space $\mathbb{R}^d$ we can define the relation $x \leq_{\text{lex}} y$ if either $x = y$ or there exists $1 \leq i \leq d$ such that $x_j = y_j$ for $j < i$ and $x_i < y_i$.

a) Show that $\leq_{\text{lex}}$ defines a total order on $\mathbb{R}^d$ (the *lexicographical order*).

b) Show that the space $(\mathbb{R}^d, \leq_{\text{lex}})$ is an ordered vector space.

c) Sketch the cone $C := \{x : 0 \leq_{\text{lex}} x\}$ in the case $d = 2$.

Solution.

a) **Reflexivity:** By definition we have $x \leq_{\text{lex}} x$, and thus $\leq_{\text{lex}}$ is reflexive.

Anti-symmetry: Assume that $x \leq_{\text{lex}} y$ and that $x \neq y$. Then there exists $i \in \{1, 2, \ldots, d\}$ such that $x_j = y_j$ for $j < i$, and $x_i < y_i$. This, however, implies that we cannot simultaneously have that $y \leq_{\text{lex}} x$ (because for that we would need that $y_i < x_i$). Therefore, the only possibility how we can have that $x \leq_{\text{lex}} y$ and $y \leq_{\text{lex}} x$ is that $y = x$.

Transitivity: Assume that $x \leq_{\text{lex}} y$ and $y \leq_{\text{lex}} z$. We have to show that $x \leq_{\text{lex}} z$. For that we can assume without loss of generality that $x \neq y$ and $y \neq z$, as else the claim is trivial. Let therefore $j \in \{1, \ldots, d\}$ be such that $x_i = y_i$ for $i < j$ and $x_j < y_j$, and let $k \in \{1, \ldots, d\}$ be such that $y_i = z_i$ for $i < k$ and $y_k < z_k$.
Then we have three possibilities: If $j < k$, then $x_i = z_i$ for $i < j$ and $x_j < y_j = z_j$. If $k < j$, then $x_i = z_i$ for $i < k$ and $x_k = y_k < z_k$. If $k = j$, then $x_i = z_i$ for $i < j$ and $x_j < y_j < z_j$. In all three cases, we obtain that $x \leq_{\text{lex}} z$.

Total order: Assume that $x, y \in \mathbb{R}^d$. We have to show that at least one of the relations $x \leq_{\text{lex}} y$ or $y \leq_{\text{lex}} x$ holds. If $x = y$, then both relations hold trivially. Else, denote by $j \in \{1, \ldots, d\}$ the smallest index for which $x_j \neq y_j$. Then $x_i = y_i$ for $i < j$, and we either have $x_j < y_j$ in which case $x \leq_{\text{lex}} y$, or $y_j < x_j$ in which case $y \leq_{\text{lex}} x$.

b) In order to show that $(\mathbb{R}^d, \leq_{\text{lex}})$ is an ordered vector space, we have to show the following two relations:

1) If $u \leq_{\text{lex}} v$ and $w \in \mathbb{R}^d$, then also $u + w \leq_{\text{lex}} v + w$.

2) If $u \leq_{\text{lex}} v$ and $\lambda > 0$, then also $\lambda u \leq_{\text{lex}} \lambda v$.

Let therefore $u, v \in \mathbb{R}^d$ with $u \leq_{\text{lex}} v$. Without loss of generality, we may assume that $u \neq v$; else both of the assertions are trivial. Thus there exists $1 \leq j \leq n$ such that $u_i = v_i$ for $i < j$ and $u_j < v_j$. Now, if $w \in \mathbb{R}^d$, then we have $u_i + w_i = v_i + w_i$ for $i < j$ and $u_j + w_j < v_j + w_j$, and thus $u + w \leq_{\text{lex}} v + w$. Moreover, if $\lambda > 0$, then $\lambda u_i = \lambda v_i$ for $i < j$ and $\lambda u_j < \lambda v_j$, and thus $\lambda u \leq_{\text{lex}} \lambda v$.

c) See Figure 1.

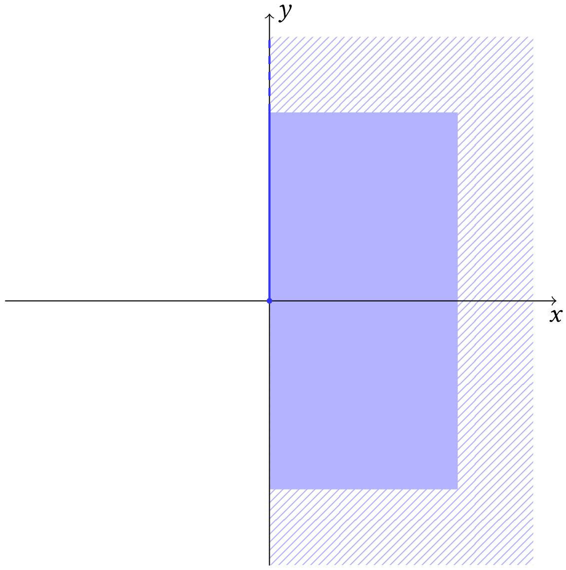

Figure 1: Sketch of the non-negative cone for the lexicographic order in $\mathbb{R}^2$. Note that the upper part of the y-axis (but not the lower part!) as well as the origin are part of the cone.

Problem 3.

Define the functions $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^2$ and $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = (x^2 - 1)^2$, and consider the multicriteria optimisation problem

$$\min_{x \in \mathbb{R}} \big(f_1(x), f_2(x)\big). \tag{1}$$

a) Sketch the image $Y := \{(f_1(x), f_2(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ of (f_1, f_2) and find all minimal points in Y.

b) Find all Pareto-optimal solutions of (1).

Solution.

a) We note first that

$$f_2(x) = (x^2 - 1)^2 = \big(f_1(x) - 1\big)^2.$$

Moreover, the image of the function $f_1(x) = x^2$ is precisely the set of non-negative real numbers. Thus we have

$$Y = \{(f_1(x), f_2(x)) : x \in \mathbb{R}\} = \big\{\big(f_1(x), (f_1(x) - 1)^2\big) : x \in \mathbb{R}\big\} = \big\{\big(y, (y - 1)^2\big) : y \geq 0\big\},$$

which is sketched in Figure 2.

b) The Pareto-optimal solutions of (1) are precisely the points $x \in \mathbb{R}$ that are mapped onto the minimal points of Y. According to the sketch in Figure 2, these are precisely the points for which $0 \leq x^2 \leq 1$, that is, the interval $[-1, 1]$.

We now look at the same problem in a more mathematical way by using the definition of Pareto-optimality. Let therefore $x \in \mathbb{R}$. We want to check whether x is a Pareto-optimum of (1).

Assume first that $|x| > 1$. Then $x^2 > 1 = f_1(1)$, and $(x^2 - 1)^2 > 0 = f_2(1)$, and thus $(f_1(1), f_2(1)) < (f_1(x), f_2(x))$. Thus no point $x \in \mathbb{R}$ with $|x| > 1$ can be a Pareto-optimum.

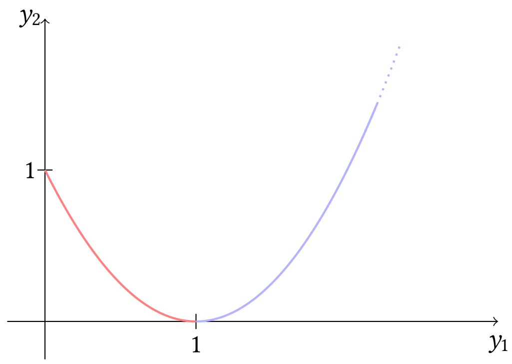

Figure 2: Image Y of the function (f_1, f_2). The red part shows the minimal points of Y.

Now assume that $|x| \leq 1$ and let $\tilde{x} \in \mathbb{R}$. If $f_1(\tilde{x}) = f_1(x)$, then also $f_2(\tilde{x}) = f_2(x)$. On the other hand, if $f_1(\tilde{x}) < f_1(x)$, that is, $\tilde{x}^2 < x^2 \leq 1$, then $(\tilde{x}^2 - 1)^2 > (x^2 - 1)^2$, that is $f_2(\tilde{x}) > f_2(x)$. Thus, for $|x| \leq 1$, there exists no $\tilde{x} \in \mathbb{R}$ such that $f_i(\tilde{x}) \leq f_i(x)$ for $i = 1, 2$, and either $f_1(\tilde{x}) < f_1(x)$ or $f_2(\tilde{x}) < f_2(x)$. This shows that every point x with $|x| \leq 1$ is Pareto-optimal.

Therefore, the set of Pareto-optimal solutions of (1) is the interval $[-1, 1]$.

Problem 4.

Define the functions $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$

$$f_1(x, y) = \frac{1}{x^4 + y^4 + 1}, \qquad f_2(x, y) = x^2 + y^2,$$

and consider the multicriteria optimisation problem

$$\min_{(x,y)\in\mathbb{R}^2} \big(f_1(x, y), f_2(x, y)\big). \tag{2}$$

a) Sketch the image $Y := \{(f_1(x, y), f_2(x, y)) : (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^2$ of (f_1, f_2) and find all minimal points in Y.

b) Show that the Pareto-optimal solutions of (2) are precisely the points of the form $(x^*, 0)$, $x^* \in \mathbb{R}$, and $(0, y^*)$, $y^* \in \mathbb{R}$.

c) Show that there does not exist any $0 \leq \lambda \leq 1$ such that $(x^*, 0) = (1/2, 0)$ is a solution of the weighted sum problem

$$\min_{(x,y)\in\mathbb{R}^2} \Big(\lambda f_1(x, y) + (1 - \lambda) f_2(x, y)\Big).$$

Solution.

a) In the next part of the problem, we will show that, for a fixed value $f_2(x, y) = R$, the function values of $f_1(x, y)$ are between $1/(1 + R^2)$ and $1/(1 + R^2/2)$. With this information, we can easily produce the sketch shown in Figure 3.

b) We note first that the point $(0, 0)$ is a Pareto-optimal solution of (2), as it is the strict global minimum of f_2.

Let now $(x^*, y^*) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ be a Pareto-optimal solution of (2), and let $R := f_2(x^*, y^*) > 0$. Then the Pareto-optimality implies that (x^*, y^*) solves the problem

$$\min_{(x,y)\in\mathbb{R}^2} f_1(x, y) \qquad \text{s.t. } f_2(x, y) = R. \tag{3}$$

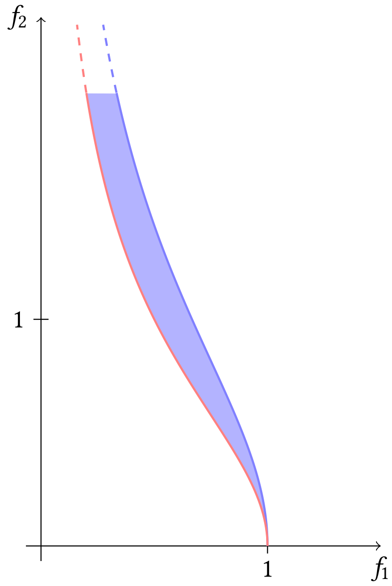

Figure 3: Sketch of the image Y of (f_1, f_2). The minimal points are shown in red.

That is, for fixed value of f_2, the point (x^*, y^*) needs to minimize the function f_1. Note that the constraint $f_2(x, y) = R$ defines a closed and bounded set, and thus a minimizer of (3) exists.

Since $R > 0$, the LICQ holds for every feasible point for (3). Thus the KKT conditions are necessary optimality conditions. In this case, they read

$$\begin{aligned} -\frac{4x^3}{(x^4 + y^4 + 1)^2} &= 2\lambda x, \\ -\frac{4y^3}{(x^4 + y^4 + 1)^2} &= 2\lambda y, \\ x^2 + y^2 &= R, \end{aligned}$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier.

It is straightforward to see that they are satisfied in the following cases:

- $x^* = \pm\sqrt{R}$ and $y^* = 0$. Here we have the function value $f_1(x^*, 0) = 1/(1 + R^2)$.
- $y^* = \pm\sqrt{R}$ and $x^* = 0$. Again, we have the function value $f_1(0, y^*) = 1/(1 + R^2)$.
- $x^* = \pm\sqrt{R/2}$ and $y^* = \pm\sqrt{R/2}$. Here we have the function value $f_1(\pm x^*, \pm y^*) = 1/(1 + R^2/2)$.

Since $\frac{1}{1+R^2} < \frac{1}{1+R^2/2}$, it follows that the solutions of (3) are obtained in the first two cases. (Note also: The problem of maximising $f_1(x, y)$ subject to the constraint $f_2(x, y) = R$ also attains a solution, and the only candidates are the KKT-points obtained in the third case above. Thus these points are the solutions of this maximisation problem.)

Until now, we have shown that all solutions of (2) are of the form asked for in the problem statement. We still have to show, though, that every such point indeed is Pareto-optimal. For that, we recall that the value of the solutions of (3) is $g(R) := 1/(1 + R^2)$ for $R \geq 0$, and g is a decreasing function for $R \geq 0$. Let therefore $(x^*, y^*) \in \mathbb{R}^2$ with $f_2(x^*, y^*) = R$ be a solution of (3) for some $R > 0$. Let moreover $(\hat{x}, \hat{y}) \in \mathbb{R}^2$ with $f_2(\hat{x}, \hat{y}) \leq R$. Then, if $f_2(\hat{x}, \hat{y}) = R$, it follows that $f_1(\hat{x}, \hat{y}) \geq f_1(x^*, y^*)$, as (x^*, y^*) solves (3). On the other hand, if $f_2(\hat{x}, \hat{y}) < R$,

$$f_1(\hat{x}, \hat{y}) \geq \inf_{f_2(x,y) = f_2(\hat{x}, \hat{y})} f_1(x, y) = \frac{1}{1 + f_1(\hat{x}, \hat{x})} > \frac{1}{1 + R^2} = f_2(x^*, y^*).$$

This shows that there exists no point $(\hat{x}, \hat{y}) \in \mathbb{R}^2$ such that $f_i(\hat{x}, \hat{y}) \le f_i(x^*, y^*)$ for $i = 1, 2$, and at least one of the inequalities $f_1(\hat{x}, \hat{y}) < f_1(x^*, y^*)$ or $f_2(\hat{x}, \hat{y}) < f_2(x^*, y^*)$ holds. Thus (x^*, y^*) is a Pareto-optimal solution of (2).

c) We first compute the gradient and Hessian of the functions f_1 and f_2. For f_1 we obtain

$$\nabla f_1(x, y) = -\frac{4}{(x^4 + y^4 + 1)^2} \begin{pmatrix} x^3 \\ y^3 \end{pmatrix}$$

and

$$H_{f_1}(x, y) = \frac{4}{(x^4 + y^4 + 1)^3} \begin{pmatrix} x^6 - 3x^2y^4 - 3x^2 & 4x^3y^3 \\ 4x^3y^3 & y^6 - 3x^4y^2 - 3y^2 \end{pmatrix}.$$

For f_2 we obtain

$$\nabla f_2(x, y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad \text{and} \qquad H_{f_2}(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Specifically, we obtain for $(x, y) = (1/2, 0)$ that

$$\nabla f_1(1/2, 0) = -\frac{128}{289} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad H_{f_1}(1/2, 0) = -\frac{256 \cdot 47}{17^3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\nabla f_2(1/2, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad H_{f_2}(1/2, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Now assume that $0 \le \lambda \le 1$ is such that $(1/2, 0)$ solves the problem $\min_{(x,y)} \lambda f_1(x, y) + (1-\lambda) f_2(x, y)$. Then the first order optimality condition needs to be satisfied at $(1/2, 0)$, that is $\lambda \nabla f_1(1/2, 0) + (1 - \lambda) \nabla f_2(1/2, 0) = 0$. Inserting the values obtained above and ignoring the second component, which is 0 anyway, we obtain the condition

$$-\frac{128}{289}\lambda + 1 - \lambda = 0.$$

By solving this equation for λ, we see that the only possibility for λ is

$$\lambda = \frac{289}{417}.$$

For this value of λ, however, we see that

$$\lambda \partial_{xx} f_1(1/2, 0) + (1 - \lambda) \partial_{xx} f_2(1/2, 0) = -\frac{2560}{2363} < 0.$$

Thus the Hessian of $\lambda f_1 + (1 - \lambda) f_2$ is not positive semi-definite, and therefore $(1/2, 0)$ is no local minimum.