Exercise #6

February 21, 2023

Problem 1.

Sketch the region $\Omega = \{(x, y) \in \mathbb{R}^2 \colon y \geq x \text{ and } y^4 \leq x^3\}$ and compute the tangent cone and the set of linearized feasible directions for each point in Ω. For which point in Ω is the LICQ satisfied?

Solution.

We first define the constraint functions,

$$c_1(x, y) = y - x \quad \text{and} \quad c_2(x, y) = x^3 - y^4,$$

so that $\Omega = \{(x, y) \in \mathbb{R}^2 : c_1(x, y) \geq 0 \text{ and } c_2(x, y) \geq 0\}$, and sketch of the region Ω is displayed in the Figure below.

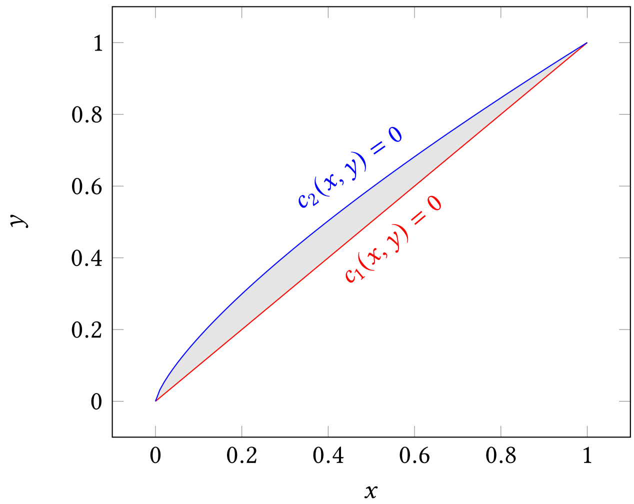

Figure 1: Region Ω in grey, with colors on the boundary specifiying the active constraints.

In order to characterise the tangent cone $T_\Omega(x, y)$ and the set of linearised feasible directions $\mathcal{F}(x, y)$, we employ Lemma 12.2 in N&W, which states that if the LICQ condition holds at a feasible point (x, y), then $T_\Omega(x, y) = \mathcal{F}(x, y)$.

Note first that the LICQ condition holds vacuously in the interior of Ω because all constraints are inactive, and therefore, $T_\Omega(x, y) = \mathcal{F}(x, y) = \mathbb{R}^2$ at interior points.

Next we consider boundary points with precisely one active constraint. Starting with points for which $c_1(x, y) = 0$, and excluding $(0, 0)$ and $(1, 1)$ where also c_2 is active. We find that $\nabla c_1(x, y) = (-1, 1)$. Since $\nabla c_1 \neq 0$, the LICQ condition holds, and so

$$\begin{aligned} T_\Omega(x,y) = \mathcal{F}(x,y) &= \left\{d = (d_1,d_2) \in \mathbb{R}^2 : \nabla c_1(x,y)^\top d \geq 0\right\} \\ &= \left\{d = (d_1,d_2) \in \mathbb{R}^2 : d_2 \geq d_1\right\}. \end{aligned}$$

Similarly, if only c_2 is active, we observe that the LICQ condition holds because $\nabla c_2(x,y) = (3x^2, -4y^3) \neq 0$ away from $(0,0)$. This yields

$$\begin{aligned} T_\Omega(x,y) = \mathcal{F}(x,y) &= \left\{d = (d_1,d_2) \in \mathbb{R}^2 : \nabla c_2(x,y)^\top d \geq 0\right\} \\ &= \left\{d = (d_1,d_2) \in \mathbb{R}^2 : 3x^2 d_1 \geq 4y^3 d_2\right\}. \end{aligned}$$

Constraint gradients at $(1,1)$ equal $\nabla c_1 = (-1,1)$ and $\nabla c_2 = (3,-4)$, which are linearly independent. Thus the LICQ condition is true, and

$$\begin{aligned} T_\Omega(1,1) = \mathcal{F}(1,1) &= \left\{d = (d_1,d_2) \in \mathbb{R}^2 : \nabla c_1(1,1)^\top d \geq 0 \text{ and } \nabla c_2(1,1)^\top d \geq 0\right\} \\ &= \left\{d = (d_1,d_2) \in \mathbb{R}^2 : d_1 \leq d_2 \text{ and } 3d_1 \geq 4d_2\right\}. \end{aligned}$$

Lastly, since $\nabla c_1(0,0) = (-1,1)$ and $\nabla c_2(0,0) = 0$, the LICQ condition fails at $(0,0)$, and we cannot expect that $T_\Omega(0,0) = \mathcal{F}(0,0)$. Readily,

$$\begin{aligned} \mathcal{F}(0,0) &= \left\{d \in \mathbb{R}^2 : \nabla c_1(0,0)^\top d \geq 0 \text{ and } \nabla c_2(0,0)^\top d \geq 0\right\} \\ &= \left\{d \in \mathbb{R}^2 : d_2 \geq d_1\right\}. \end{aligned}$$

In order to find the tangent cone, we first consider limiting directions along the constraint boundaries $c_1(x,y) = 0$ and $c_2(x,y) = 0$ as $(x,y) \to (0,0)$. Travelling towards $(0,0)$ when c_1 is active, we may put, using the notation in N&W,

$$z_k = (1/k, 1/k) \quad \text{and} \quad t_k = 1/k,$$

and obtain the limiting direction

$$d = \lim_{k\to\infty} \frac{z_k - (0,0)}{t_k} = (1,1).$$

Note: the length of d is irrelevant; we only care about its direction. Similarly, travelling along $c_2(x,y) = 0$ yields $d = (0,1)$, using for example, the sequences

$$z_k = \left(k^{-1/3}, k^{-1/4}\right) \quad \text{and} \quad t_k = k^{-1/4}.$$

It can furthermore be seen that approaching $(0,0)$ from the interior of Ω gives tangent directions "between" these borderline cases, and so

$$T_\Omega(0,0) = \left\{d \in \mathbb{R}^2 : d_2 \geq d_1 \geq 0\right\}.$$

Problem 2.

Assume that one wants to solve the optimisation problem

$$\max_x f(x) \qquad \text{such that} \qquad \begin{cases} c_i(x) = 0 & \text{for all } i \in \mathcal{E}, \\ c_i(x) \geq 0 & \text{for all } i \in \mathcal{I}. \end{cases}$$

How can we modify the KKT conditions such that one obtains (first order) necessary conditions for this maximisation problem?

Solution.

Have already discussed in the class!

Problem 3.

Consider the constrained optimization problem

$$\min_{(x,y)} (x^2 + y^2) \qquad \text{such that} \qquad \begin{cases} x + y \geq 1, \\ \quad y \leq 2, \\ \quad y^2 \geq x. \end{cases}$$

a) Formulate the KKT-conditions for this optimization problem.

b) Find all KKT points for this optimization problem.

c) Find all local and global minima for this optimization problem.

(Part b) can be very tedious. One strategy is to consider all possible active sets and determine for each active set whether KKT-points exist. It can also be extremely helpful to sketch the feasible set and the function.)

Solution.

a) We begin by stating the problem in standard form, writing $\mathbf{x} = [x, y]^T$:

$$\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) \quad \text{s.t.} \quad c_i(\mathbf{x}) \geq 0, \quad i = 1, 2, 3,$$

where

$$\begin{aligned} f(\mathbf{x}) &= x^2 + y^2, \\ c_1(\mathbf{x}) &= x + y - 1, \\ c_2(\mathbf{x}) &= 2 - y, \\ c_3(\mathbf{x}) &= y^2 - x. \end{aligned}$$

The KKT conditions can now be stated as follows:

$$2x^* - \lambda_1^* + \lambda_3^* = 0 \tag{1a}$$
$$2y^* - \lambda_1^* + \lambda_2^* - 2y^*\lambda_3^* = 0 \tag{1b}$$
$$x^* + y^* - 1 \geq 0 \tag{1c}$$
$$2 - y^* \geq 0 \tag{1d}$$
$$y^{*2} - x^* \geq 0 \tag{1e}$$
$$\lambda_i^* \geq 0, \quad i = 1, 2, 3 \tag{1f}$$
$$\lambda_1^*(x^* + y^* - 1) = 0 \tag{1g}$$
$$\lambda_2^*(2 - y^*) = 0 \tag{1h}$$
$$\lambda_3^*(y^{*2} - x^*) = 0. \tag{1i}$$

b) The feasible set is sketched in Figure 2.

We will find all KKT points by systematically considering all possible active sets of constraints. Remember that a constraint c_i is active at a point $\mathbf{x}$ if $c_i(\mathbf{x}) = 0$. Also, the LICQ conditions are satisfied at every point we consider here; with one active constraint, the LICQ conditions hold trivially, since the gradient ∇c_i never vanishes. Moreover, in the cases with two constraints it is not hard to check that the LICQ conditions do hold.

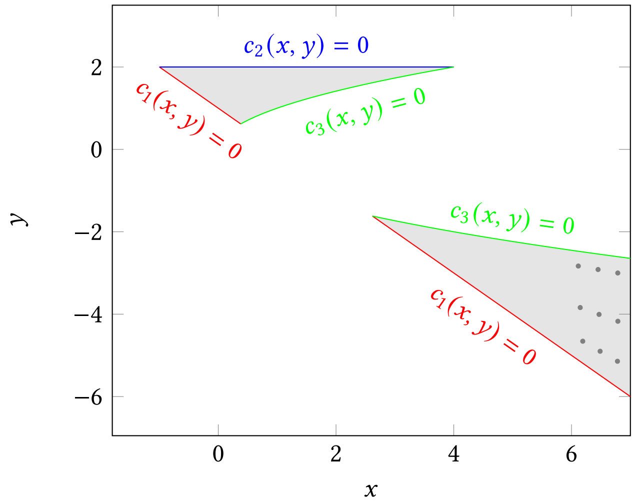

Figure 2: Feasible set. Note: The lower "triangle" extends further toward infinity.

Observe that if $\mathbf{x}^* = [x^*, y^*]^T$ is a KKT point, then from (1a) and (1b) we have:

$$x^* = \frac{\lambda_1^* - \lambda_3^*}{2}, \qquad y^* = \frac{\lambda_1^* - \lambda_2^*}{2(1 - \lambda_3^*)}.$$

From here on, we will drop the asterisk in the notation and write x for x^*, etc.

First, suppose that the active set is empty, i.e. neither of (1c)-(1e) are equalities. This corresponds to the interior of the domain. Then, by (1g)-(1i), we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and so $x = y = 0$. But this point is not feasible, since it violates condition (1c). Thus, with the active set empty, there are no KKT points.

Next, we consider the case when the active set contains one index, i.e. exactly one of (1c)-(1e) is an equality. This corresponds to the boundaries of the domain, excepting the corner points. If (1c) is active, then $\lambda_2 = \lambda_3 = 0$ while $\lambda_1 \geq 0$. We get

$$x = \frac{\lambda_1}{2}, \qquad y = \frac{\lambda_1}{2},$$

and inserting this into (1c) (which is now an equality), we get the condition

$$\frac{\lambda_1}{2} + \frac{\lambda_1}{2} - 1 = 0 \Rightarrow \lambda_1 = 1,$$

giving us the point $(x, y) = (\frac{1}{2}, \frac{1}{2})$. But this point violates condition (1e), so $(\frac{1}{2}, \frac{1}{2})$ is not a KKT point.

If (1d) is active, then $\lambda_1 = \lambda_3 = 0$ while $\lambda_2 \geq 0$, so

$$x = 0, \qquad y = -\frac{\lambda_2}{2}.$$

Inserting this into the equality (1d), we get

$$2 + \frac{\lambda_2}{2} = 0 \Rightarrow \lambda_2 = -4.$$

Since the Lagrange multiplier is negative, KKT conditions are not satisfied at this point.

If (1e) is active, then $\lambda_1 = \lambda_2 = 0$ while $\lambda_3 \geq 0$, so

$$x = -\frac{\lambda_3}{2}, \qquad y = 0.$$

Inserting this into the equality (1e), we get

$$\frac{\lambda_3}{2} = 0 \Rightarrow \lambda_3 = 0.$$

This gives the candidate point $(0, 0)$, which is not feasible since it violates (1c), and thereby is not a KKT point.

Having considered all possible active sets of one index, we now turn to the cases with two indices, i.e. exactly two of (1c)-(1e) are equalities. This corresponds to the corner points of the domain. First, if (1c) and (1d) are both active, then $\lambda_3 = 0$ while $\lambda_1, \lambda_2 \geq 0$. This gives us

$$x = \frac{\lambda_1}{2}, \quad y = \frac{\lambda_1 - \lambda_2}{2}.$$

Plugging this into equalities (1c) and (1d) yields:

$$\begin{aligned} \frac{\lambda_1}{2} + \frac{\lambda_1 - \lambda_2}{2} - 1 &= 0 \\ 2 - \frac{\lambda_1 - \lambda_2}{2} &= 0, \end{aligned}$$

with solutions $\lambda_1 = -2$ and $\lambda_2 = -6$. Since the multipliers are negative, this is not a KKT point.

Next, if (1c) and (1e) are both active, then $\lambda_2 = 0$ while $\lambda_1, \lambda_3 \geq 0$, which means

$$x = \frac{\lambda_1 - \lambda_3}{2}, \quad y = \frac{\lambda_1}{2(1 - \lambda_3)}.$$

Plugging this into equalities (1c) and (1e) yields:

$$\begin{aligned} \frac{\lambda_1 - \lambda_3}{2} + \frac{\lambda_1}{2(1 - \lambda_3)} - 1 &= 0 \\ \frac{\lambda_1^2}{4(1 - \lambda_3)^2} - \frac{\lambda_1 - \lambda_3}{2} &= 0. \end{aligned}$$

Solving this set of equations yields $\lambda_1 = 5 \pm \frac{9}{\sqrt{5}}$ and $\lambda_3 = 2 \pm \frac{4}{\sqrt{5}}$, thereby giving the candidate points $(x, y) = (\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-1 \mp \sqrt{5}))$ which both satisfy the KKT conditions. Since $\lambda_1, \lambda_3 \geq 0$, these points are minimizer candidates. Note: This result can be arrived upon by the easier approach of first finding the points (x, y) where c_1 and c_3 are both active, then working out what λ_1 and λ_3 are.

Finally, we check the case where (1d) and (1e) are both active, i.e. $\lambda_1 = 0$ while $\lambda_2, \lambda_3 \geq 0$. This gives us

$$x = -\frac{\lambda_3}{2}, \quad y = -\frac{\lambda_2}{2(1 - \lambda_3)}.$$

Plugging this into equalities (1d) and (1e) yields:

$$\begin{aligned} 2 + \frac{\lambda_2}{2(1 - \lambda_3)} &= 0 \\ \frac{\lambda_2^2}{4(1 - \lambda_3)^2} + \frac{\lambda_3}{2} &= 0, \end{aligned}$$

which can be solved to find $\lambda_2 = -28$ and $\lambda_3 = -8$. Since the multipliers are negative, this is not a KKT point.

Concerning the case with all constraints active, we may conclude that no KKT point exists; all three constraint functions cannot be active at the same point. The investigation is summarized in the table below.

Point	λ_1	λ_2	λ_3	KKT?
(0,2)	0	-4	0	No
$(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$	$5 + \frac{9}{\sqrt{5}}$	0	$2 + \frac{4}{\sqrt{5}}$	Yes
$(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5}))$	$5 - \frac{9}{\sqrt{5}}$	0	$2 - \frac{4}{\sqrt{5}}$	Yes
(-1,2)	-2	-6	0	No
(4,2)	0	-28	-8	No

c) To determine whether the KKT points that are minimizer candidates are in fact local minimizers, we check the second order sufficient conditions from Theorem 12.6 in N&W, i.e. whether

$$w^T \nabla^2_{xx} \mathcal{L}(x, \lambda) w > 0 \ \forall w \in C(x, \lambda), w \neq 0, \tag{2}$$

where, $C(x, \lambda)$ is the critical cone at x, given by (12.53) in N&W.

For both candidates, i.e. $(\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-1 \mp \sqrt{5}))$, we have that the critical cone is simply given as $C(x, \lambda) = \{0\}$. This is because any $w \in C(x, \lambda)$ must be orthogonal to the $\nabla c_i(x)$ for which $\lambda_i > 0$, of which there are two for each point. Since the LICQ conditions hold at both points, these two vectors are linearly independent and thus span $\mathbb{R}^2$. The only vector orthogonal to $\mathbb{R}^2$ is the zero vector. Thereby, the only vector in $C(x, \lambda)$ is the zero vector for these points, and thus condition (2) holds. We can conclude that $(\frac{1}{2}(3\pm\sqrt{5}), \frac{1}{2}(-1\mp\sqrt{5}))$ are strict local minimizers.

We note that $f(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})) < f(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$ and $f(\mathbf{x}) \to \infty$ in the unbounded region of the feasible domain. This means that $(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5}))$ is a global minimizer and $(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$ is a local minimizer.

Problem 4.

Consider the constrained optimization problem

$$\min_{(x,y)} (xy) \qquad \text{such that} \qquad \begin{cases} y \geq x, \\ y^4 \leq x^3. \end{cases}$$

(Note that the constraint set is the same as in Problem 1.)

a) Find all KKT points and local minima for this optimization problem.

b) Compute the critical cone at $(0, 0)$ as defined in the lecture and Nocedal & Wright, and show that there exist directions p contained in the critical cone for which $p^T \nabla^2 \mathcal{L}((0, 0), \lambda^*) p < 0$.

c) Show that $p^T \nabla^2 \mathcal{L}((0, 0), \lambda^*) p \geq 0$ for all vectors p contained in the tangent cone to the feasible set at $(0, 0)$.

Solution.

Have already discussed in the class!