Problem 1.

Consider the unconstrained optimisation problem

$$\min_{x,y} f(x, y),$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x, y) = \frac{1}{4}x^4 + 2x^2y^2 + 4x^2y + 2x^2 + y^2 + 2y.$$

a) Find all local and global minimisers of f.
(10 points)

b) Determine whether the function f is convex.
(5 points)

c) Assume that you want to perform one step of the gradient descent method starting from the point $(x, y) = (0, 0)$. For which step lengths $\alpha > 0$ are the strong Wolfe conditions with parameters $c_1 = 0.1$ and $c_2 = 0.9$ satisfied?
(10 points)

d) Assume you want to solve this optimisation problem with Newton's method, using backtracking Armijo line search. Can you guarantee that this method converges for all initialisations? In case the algorithm converges, what convergence rate do you expect?
(10 points)

Solution.

a) We start by computing the gradient of f: We have

$$\nabla f(x, y) = \begin{pmatrix} x^3 + 4xy^2 + 8xy + 4x \\ 4x^2y + 4x^2 + 2y + 2 \end{pmatrix}.$$

The first order optimality condition $\nabla f(x, y)$ thus becomes

$$\begin{aligned} x^3 + 4xy^2 + 8xy + 4x &= 0, \\ 4x^2y + 4x^2 + 2y + 2 &= 0. \end{aligned}$$

The first equation simplifies to

$$x\left(x^2 + 4y^2 + 8y + 4\right) = 0,$$

which holds if either $x = 0$ or $x^2 + 4y^2 + 8y + 4 = 0$. In the first case, the second equation implies that $y = -1$. Moreover, since we can write

$$x^2 + 4y^2 + 8y + 4 = x^2 + 4(y + 1)^2,$$

we obtain for the latter case, again, that $x = 0$ and $y = -1$. Thus the only critical point of f is the point $(x^*, y^*) = (0, -1)$. This is thus the only candidate for a local or global minimum.

In order to show that this is in fact a global minimum, we show that f is coercive. To that end, we note that

$$f(x, y) = \frac{1}{4}x^4 + 2(xy + x)^2 + (y + 1)^2 - 1 \geq \frac{1}{4}x^4 + (y + 1)^2 - 1.$$

The right hand side of this expression clearly tends to $+\infty$ if either $x \to \pm\infty$ or $y \to \pm\infty$, which shows that f is coercive. Thus f attains a global minimum.

As shown above, the only candidate for this global minimum is the point $(0, 1)$, which shows that is the unique global minimum of f. Moreover, there exist no other local minimisers.

b) For checking whether f is convex or not, we compute the Hessian of f. We have

$$H_f(x, y) = \begin{pmatrix} 3x^2 + 4y^2 + 8y + 4 & 8xy + 8x \\ 8xy + 8x & 4x^2 + 2 \end{pmatrix}.$$

Specifically we obtain for $y = 0$ that

$$H_f(x, 0) = \begin{pmatrix} 3x^2 + 4 & 8x \\ 8x & 4x^2 + 2 \end{pmatrix}.$$

For $x = 1$, this further becomes

$$H_f(1, 0) = \begin{pmatrix} 7 & 8 \\ 8 & 6 \end{pmatrix},$$

which is an indefinite matrix, as $\det H_f(1, 0) = 7 \cdot 6 - 8^2 = -22 < 0$. This shows that f is not convex.

c) At the point $(0, 0)$ we have

$$\nabla f(0, 0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

The search direction for the line search is therefore $p = (0, -2)$, and we have $\langle p, \nabla f(0, 0)\rangle = -4$. The Armijo condition thus requires that

$$f(0, -2\alpha) \leq f(0, 0) - 4c_1\alpha,$$

which, concretely, becomes

$$4\alpha^2 - 4\alpha \le -4c_1\alpha,$$

or

$$\alpha \le 1 - c_1.$$

With $c_1 = 0.1$, this means that

$$\alpha \le \frac{9}{10}.$$

For the strong curvature condition, we need to compute

$$\langle \nabla f(0, -2\alpha), p\rangle = \langle (0, 2 - 4\alpha), (0, -2)\rangle = 8\alpha - 4.$$

The condition thus becomes

$$\left|8\alpha - 4\right| \le 4c_2,$$

which, with $c_2 = 0.9$, simplifies to

$$\left|\alpha - 1/2\right| \le \frac{c_2}{2} = 9/20.$$

For the strong curvature condition, the step length $\alpha > 0$ is thus admissible if $1/20 \le \alpha \le 19/20$.

Combining the two restrictions, we obtain that the strong Wolfe conditions are satisfied for

$$\frac{1}{20} \le \alpha \le \frac{9}{10}.$$

d) Important things to consider here are:

- A minimum requirement for any line search method to work at all is that the search direction is in every step a descent direction. For Newton's method, this can be guaranteed if the Hessian is in all points positive definite.
- Newton's method with backtracking converges (in the sense that $\nabla f \to 0$), provided that the eigenvalues of the Hessian are uniformly bounded below over all the iterations and the function is coercive.
- Generally, we can expect quadratic convergence of Newton's method, if the Hessian at the point we are converging to is non-singular.

For the concrete problem in question, we have shown that the function is not convex. Thus it can happen for some initialisations that Newton's method does not work at all, because the backtracking steps break down. In addition, the Hessian at the unique local/global minimum is

$$H_f(0, -1) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

which is obviously singular. Thus we cannot expect quadratic convergence, but typically only linear convergence.

Problem 2.

Consider the optimisation problem

$$x + y \longrightarrow \min$$

subject to the constraint $(x, y) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is given by the constraints

$$x^2 + y^2 \geq 25, \qquad y \geq 0, \qquad x \leq 5, \quad \text{and} \quad 3y \leq 4x.$$

a) Sketch the set Ω and determine for each point in Ω whether the LICQ holds.
(5 points)

b) Determine the tangent cone and the cone of linearised feasible directions to the set Ω in the points $(x, y) = (3, 4)$ and $(x, y) = (5, 0)$.
(10 points)

c) Find all KKT points and all local and global minimisers for this optimisation problem.
(15 points)

Solution.

For the solution of this problem, we define

$$\begin{aligned} f(x, y) &= x + y, \\ c_1(x, y) &= x^2 + y^2 - 25, \\ c_2(x, y) &= y, \\ c_3(x, y) &= 5 - x, \\ c_4(x, y) &= 4x - 3y. \end{aligned}$$

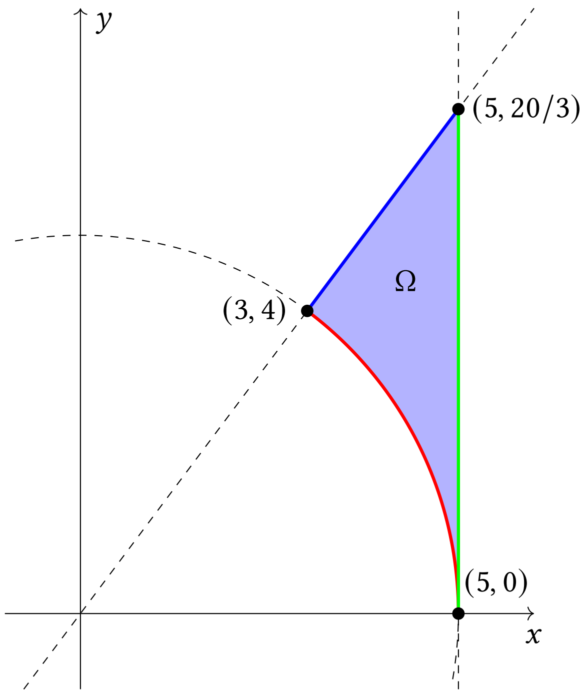
Then we have the constrained optimisation problem

$$\min_{x,y} f(x, y) \qquad \text{subject to } c_i(x, y) \geq 0 \text{ for } i = 1, 2, 3, 4.$$

Moreover, we have

$$\nabla f(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \nabla c_1(x, y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}, \quad \nabla c_2(x, y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \nabla c_3(x, y) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \nabla c_4(x, y) = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

a) Here is a sketch of the set Ω:



The red line is the set where the constraint c_1 is active; the green line is the set where the constraint c_3 is active; the blue line is the set where the constraint c_4 is active.

From the sketch, we see that we have the following possibilities for the active constraints:

- No constraints are active. In this case the LICQ trivially holds.
- Only the constraint c_1 is active. Since $\nabla c_1(x, y) \neq 0$ unless $(x, y) = (0, 0)$, which is an infeasible point, it follows that LICQ holds in this case.
- Only the constraint c_3 is active. Since $\nabla c_3(x, y)$ can never be zero, it follows that LICQ holds.
- Only the constraint c_4 is active. Since $\nabla c_4(x, y)$ can never be zero, it follows that LICQ holds.
- The constraints c_1 and c_4 are active, that is, $(x, y) = (3, 4)$. Here

$$\nabla c_1(3, 4) = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \qquad \text{and} \qquad \nabla c_4(3, 4) = \begin{pmatrix} 4 \\ -3 \end{pmatrix},$$

  which are obviously linearly independent. Thus LICQ holds.
- The constraints c_3 and c_4 are active, that is, $(x, y) = (5, 20/3)$. Since ∇c_3 and ∇c_4 are linearly independent, LICQ holds.

- The constraints c_1, c_2, and c_3 are active, that is, $(x, y) = (5, 0)$. Since we have three active constraints in two dimensions, LICQ cannot hold. Explicitly, we see that $\nabla c_1(5, 0) = -10\nabla c_3(5, 0)$.

To summarise, LICQ holds in every point apart from $(5, 0)$.

b) At $(x, y) = (3, 4)$, the constraints c_1 and c_4 are active, and LICQ holds. Thus the tangent cone and the cone of linearised feasible directions are the same, and

$$T_\Omega(3, 4) = \mathcal{F}_\Omega(3, 4) = \{(p, q) \in \mathbb{R}^2 : 3p + 4q \geq 0 \text{ and } 4p - 3q \leq 0\}.$$

At $(5, 0)$ the constraints c_1, c_2, and c_3 are active. However, LICQ does not hold, so we cannot assume a-priori that $T_\Omega(5, 0) = \mathcal{F}_\Omega(5, 0)$. For the cone of linearised feasible directions we obtain

$$\mathcal{F}_\Omega(5, 0) = \{(p, q) \in \mathbb{R}^2 : 2p \geq 0,\ q \geq 0,\ -p \geq 0\} = \{(0, q) \in \mathbb{R}^2 : q \geq 0\}.$$

For the computation of $T_\Omega(5, 0)$ we note first that, necessarily, the inclusion $T_\Omega(5, 0) \subset \mathcal{F}_\Omega(5, 0)$ holds.

Now let $q \geq 0$ and define the sequences

$$z_k = (5, q/k) \qquad \text{and} \qquad t_k = 1/k.$$

Then $z_k \in \Omega$ for all sufficiently large k, $z_k \to (5, 0)$ as $k \to \infty$, and $t_k \to 0$ as $k \to \infty$. Thus

$$(0, q) = \lim_{k\to\infty} \frac{z_k - (5, 0)}{t_k} \in T_\Omega(5, 0).$$

This shows that

$$\mathcal{F}_\Omega(5, 0) = \{(0, q) : q \geq 0\} \subset T_\Omega(5, 0).$$

Since we already have seen that $T_\Omega(5, 0) \subset \mathcal{F}_\Omega(5, 0)$, we can conclude that

$$T_\Omega(5, 0) = \mathcal{F}_\Omega(5, 0) = \{(0, q) : q \geq 0\}.$$

c) The KKT conditions for this problem read

$$\begin{aligned}
1 - 2x\lambda_1 + \lambda_3 - 4\lambda_4 &= 0,\\
1 - 2y\lambda_1 - \lambda_2 + 3\lambda_4 &= 0,\\
x^2 + y^2 \geq 25, \quad \lambda_1(x^2 + y^2 - 25) &= 0,\\
y \geq 0, \quad \lambda_2 y &= 0,\\
x \leq 5, \quad \lambda_3(5 - x) &= 0,\\
3y \leq 4x, \quad \lambda_4(4x - 3y) &= 0,\\
\lambda_i \geq 0, \quad i &= 1, 2, 3, 4.
\end{aligned}$$

We now consider the different possibilities for the active set, as we did in part a).

- No constraints are active. Since $\nabla f(x,y) = (1,1) \neq (0,0)$ for any (x,y), we have no KKT points in this case.
- Only the constraint c_1 is active. Here we obtain the conditions

$$1 = 2x\lambda_1, \qquad 1 = 2y\lambda_1, \qquad x^2 + y^2 = 25.$$

  Within the feasible set, we obtain the solution $x = y = 5/\sqrt{2}$ with corresponding Lagrange multiplier $\lambda_1 = 1/(5\sqrt{2}) \geq 0$. Thus the KKT conditions hold at this point.

  In order to determine whether this may be a possible local minimum, we consider the second order necessary condition. To that end, we compute the Hessian

$$H_f(5/\sqrt{2}, 5/\sqrt{5}) - \frac{1}{5\sqrt{2}} H_{c_1}(5/\sqrt{2}, 5/\sqrt{2}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  Since this matrix is negative definite on the critical cone

$$C(5/\sqrt{2}, 5/\sqrt{2}) = \{(p,q) \in \mathbb{R}^2 : p + q = 0\}$$

  (and LICQ holds at the point under consideration), this point cannot be a local minimum.
- Only the constraint c_3 is active. Here we obtain the equations

$$1 + \lambda_3 = 0, \qquad 1 = 0, \qquad x = 5,$$

  which are obviously not solvable. Thus no KKT point with this active set exists.
- Only the constraint c_4 is active. Here the KKT conditions become

$$1 - 4\lambda_4 = 0, \qquad 1 + 3\lambda_4 = 0, \qquad 4x = 3y.$$

  We see that the first two equations cannot hold simultaneously, and thus no KKT point with this active set exists.
- The constraints c_1 and c_4 are active and $(x,y) = (3,4)$. Here we obtain the conditions

$$1 - 6\lambda_1 - 4\lambda_4 = 0 \qquad \text{and} \qquad 1 - 8\lambda_1 + 3\lambda_4 = 0$$

  for the Lagrange parameters. Solving this system yields the Lagrange parameters $\lambda_1 = 7/50$ and $\lambda_4 = 1/25$, both of which are strictly positive. Since the gradients $\nabla c_1(3,4)$ and $\nabla c_4(3,4)$ are linearly independent and the corresponding Lagrange parameters are strictly positive, it follows that the critical cone at $(3,4)$ is equal to $\{(0,0)\}$, and thus the second order sufficient optimality condition holds trivially. Thus this point is a strict local minimum.

- The constraints c_3 and c_4 are active and $(x, y) = (5, 20/3)$. Here we obtain the conditions

$$1 + \lambda_3 - 4\lambda_4 = 0 \qquad \text{and} \qquad 1 + 3\lambda_4 = 0.$$

  We see immediately that the second equation implies that $\lambda_4 = -1/3 < 0$. Thus this point is no KKT point.

- The constraints c_1, c_2, and c_3 are active and $(x, y) = (5, 0)$. Here we obtain the conditions

$$1 - 10\lambda_1 + \lambda_3 = 0 \qquad \text{and} \qquad 1 - \lambda_2 = 0.$$

  One possible solution of this equation is $\lambda_1 = 1/10$, $\lambda_2 = 1$, $\lambda_3 = 0$. With these parameters, the KKT conditions are satisfied, and thus this point is a KKT point. Since the gradients $\nabla c_1(5, 0) = (10, 0)$ and $\nabla c_2(5, 0) = (0, 1)$ are linearly independent and the corresponding Lagrange parameters are strictly positive, it follows that the critical cone is equal to $\{0, 0\}$, and thus the second order sufficient optimality condition holds trivially. Thus this point is a strict local minimum.

To summarise, we have found the KKT points $(5/\sqrt{2}, 5/\sqrt{2})$, $(3, 4)$, and $(5, 0)$. Moreover, the points $(3, 4)$ and $(5, 0)$ were local minima.

Finally, we have to decide whether one of the points $(3, 4)$ or $(5, 0)$ is a global minimum. For this, we note first that the set Ω is closed and bounded, and thus the problem actually admits a global minimum. Moreover, we see that $f(3, 4) = 7$ whereas $f(5, 0) = 5$. Thus the point $(5, 0)$ is the (unique) global minimum of this problem.

Problem 3.

Let $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$ be given, and consider the optimisation problem

$$\|x\|_1 \to \min \qquad\qquad \text{subject to } Ax \geq b. \tag{1}$$

Formulate the Lagrangian dual of (1) as a constrained optimisation problem.
(10 points)

Solution.

The Lagrangian of this problem is defined as

$$\mathcal{L}(x, \lambda) = \|x\|_1 - \langle \lambda, Ax - b\rangle.$$

The dual cost function thus becomes

$$q(\lambda) = \min_{x\in\mathbb{R}^d}\big(\|x\|_1 - \langle \lambda, Ax - b\rangle\big) = \langle \lambda, b\rangle + \min_{x\in\mathbb{R}^d}\big(\|x\|_1 - \langle A^T\lambda, x\rangle\big).$$

Moreover, we can simplify the minimisation problem in this expression to

$$\min_{x\in\mathbb{R}^d}\big(\|x\|_1 - \langle A^T\lambda, x\rangle\big) = \min_{x\in\mathbb{R}^d}\Big(\sum_{i=1}^{d}\big(|x_i| - (A^T\lambda)_i x_i\big)\Big) = \sum_{i=1}^{d}\min_{t\in\mathbb{R}}\big(|t| - (A^T\lambda)_i t\big).$$

If $|(A^T\lambda)_i| > 1$, then the term $|t| - (A^T\lambda)_i t$ is unbounded below. On the other hand, if $|(A^T\lambda)_i| \le 1$, then $|t| - (A^T\lambda)_i t \ge 0$ for all $t \in \mathbb{R}$, and we have equality for $t = 0$. Thus

$$\min_{t\in\mathbb{R}}\big(|t| - (A^T\lambda)_i t\big) = \begin{cases} 0 & \text{if } |(A^T\lambda)_i| \le 1, \\ -\infty & \text{if } |(A^T\lambda)_i| > 1. \end{cases}$$

Summing up over all these terms, we obtain that

$$q(\lambda) = \begin{cases} \langle \lambda, b\rangle & \text{if } \|A^T\lambda\|_\infty \le 1, \\ -\infty & \text{if } \|A^t\lambda\|_\infty > 1. \end{cases}$$

The dual problem thus becomes

$$\max_{\lambda\ge 0} q(\lambda),$$

which can be written as the constrained optimisation problem

$$\langle \lambda, b\rangle \to \max \qquad \text{subject to} \begin{cases} \|A^T\lambda_\infty \le 1, \\ \lambda \ge 0. \end{cases}$$

Problem 4.

Let $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$ be given, and assume that $f : \mathbb{R}^d \to \mathbb{R}$ is strictly convex, coercive, and continuously differentiable. Assume moreover that the equation $Ax = b$ has some solution $\hat{x} \in \mathbb{R}^d$. We consider the constrained optimisation problem

$$\min_x f(x) \qquad \text{subject to } Ax = b. \tag{2}$$

Moreover, for $\lambda \in \mathbb{R}^m$ and $\mu > 0$, we denote by

$$\mathcal{L}_A(x, \lambda; \mu) = f(x) - \langle \lambda, Ax - b\rangle + \frac{\mu}{2}\|Ax - b\|_2^2$$

the augmented Lagrangian of this problem.

a) Show that the constrained optimisation problem (2) admits a unique solution $x^* \in \mathbb{R}^d$.
(5 points)

b) Show that the optimisation problem

$$\min_{x\in\mathbb{R}^d} \mathcal{L}_A(x, \lambda; \mu) \tag{3}$$

has for every $\lambda \in \mathbb{R}^m$ and every $\mu > 0$ a unique solution $x_{\lambda,\mu}$.
(5 points)

c) Show that $x_{\lambda,\mu} = x^*$, if and only if $Ax_{\lambda,\mu} = b$.
(5 points)

Solution.

a) The set $\Omega := \{x \in \mathbb{R}^d : Ax = b\}$ is non-empty (by assumption) and closed, and f is coercive. Thus, the minimisation problem (2) admits a solution. Since Ω is convex (being an affine set) and f is strictly convex, the solution is unique.

b) We can estimate

$$\begin{aligned}
\mathcal{L}_A(x,\lambda;\mu) &= f(x) - \langle \lambda, Ax-b\rangle + \frac{\mu}{2}\|Ax-b\|_2^2 \\
&\ge f(x) - \|\lambda\|_2\|Ax-b\|_2 + \frac{\mu}{2}\|Ax-b\|_2^2 \\
&= f(x) + \frac{\mu}{2}\Big(\|Ax-b\|_2^2 - 2\frac{\|\lambda\|_2}{\mu}\|Ax-b\|_2\Big) \\
&= f(x) + \frac{\mu}{2}\Big(\|Ax-b\|_2 - \frac{\|\lambda\|_2}{\mu}\Big)^2 - \frac{\mu}{2}\frac{\|\lambda\|_2^2}{\mu^2} \\
&\ge f(x) - \frac{\|\lambda\|_2^2}{2\mu}.
\end{aligned}$$

Since the function f is coercive, it follows that $\mathcal{L}_A(\cdot,\lambda;\mu)$ is coercive as well, as it is, up to a constant, bounded below by f. Thus $\mathcal{L}_A(\cdot,\lambda;\mu)$ has a global minimiser. Moreover, $\mathcal{L}_A(x,\lambda;\mu)$ is strictly convex, as it is the sum of the strictly convex function f, the linear function $x \mapsto -\langle\lambda, Ax-b\rangle$, and the convex function $x \mapsto \frac{\mu}{2}\|Ax-b\|_2^2$. Thus the minimiser of $\mathcal{L}_A(\cdot,\lambda;\mu)$ is unique.

c) The "only if" part is clear: If $x_{\lambda,\mu}$ solves (2), then $x_{\lambda,\mu}$ in particular has to be feasible, which means that $Ax_{\lambda,\mu} = b$.

Now assume that $Ax_{\lambda,\mu} = b$. Since $x_{\lambda,\mu}$ is a minimiser of $\mathcal{L}_A(x,\lambda;\mu)$, it follows that $\nabla_x\mathcal{L}_A(x,\lambda;\mu) = 0$, that is,

$$\nabla f(x_{\lambda,\mu}) - A^T\lambda + \mu A^T(Ax_{\lambda,\mu} - b) = 0.$$

Since $Ax_{\lambda,\mu} = b$, this simplifies to

$$\nabla f(x_{\lambda,\mu}) = A^T\lambda.$$

Thus $x_{\lambda,\mu}$ satisfies the conditions

$$\nabla f(x_{\lambda,\mu}) = A^T\lambda \qquad \text{and} \qquad Ax_{\lambda,\mu} = b,$$

which are precisely the KKT conditions for the problem (2). Since the function f is convex and the constraints are linear, the KKT conditions are sufficient optimality conditions, and thus $x_{\lambda,\mu}$ solves (2), that is, $x_{\lambda,\mu} = x^*$.

Problem 5.

Define the functions $f_1, f_2 \colon \mathbb{R}_{>0} \to \mathbb{R}$,

$$f_1(x) = (x-1)^2, \qquad f_2(x) = -\ln(x).$$

Find all Pareto-optimal solutions of the multi-criteria optimisation problem

$$\min_{x>0} \big(f_1(x), f_2(x)\big).$$

(10 points)

Solution.

A point $x^* > 0$ is Pareto-optimal, if there does not exist any point $\hat{x} > 0$ with $\hat{x} \neq x^*$ such that either $f_1(\hat{x}) < f_1(x^*)$ and $f_2(\hat{x}) \le f_2(x^*)$, of $f_1(\hat{x}) \le f_1(x^*)$ and $f_2(\hat{x}) < f_2(x^*)$.

Since the function $f_2(x) = -\ln(x)$ is stricty decreasing for $x > 0$, the conditions $\hat{x} \neq x^*$ and $f_2(\hat{x}) \le f_2(x^*)$ are equivalent to $\hat{x} > x^*$. Thus x^* is Pareto-optimal, if and only if there does not exist $\hat{x} > x^*$ with $f_1(\hat{x}) \le f_1(x^*)$.

Now we note that the function $f_1(x) = (x-1)^2$ is strictly decreasing for $x < 1$, strictly increasing for $x > 1$, and has a (unique) global minimum at $x = 1$. Thus no point $x < 1$ can be a Pareto-optimal solution. (For every $x < 1$ we have $f_1(x) > f_1(1)$ and $f_2(x) > f_2(1)$.) On the other hand, if $x \ge 1$, then there does not exist any $\hat{x} > x$ such that $f_1(\hat{x}) \le f_1(x)$, because f_1 is strictly increasing for $x \ge 1$.

This shows that all points $x^* \ge 1$ are Pareto-optimal solutions of the multi-criteria optimisation problem.

Note: Because f_1 and f_2 are strictly convex, it is also possible to find all the Pareto-optimal solutions by looking at the weighted sum problems $\min_{x>0}\big(\lambda f_1(x) + (1-\lambda) f_2(x)\big)$ for $0 \le \lambda \le 1$. The solutions of these problems will be precisely the Pareto-optimal solutions of the multi-criteria optimisation problem. Moreover, since f_1 and f_2 are strictly convex, these can simply be found by solving the equations $\lambda f_1'(x) + (1-\lambda) f_2'(x) = 0$, or $2\lambda(x-1) - (1-\lambda)/x = 0$, each of which is a quadratic equation with a single positive solution (apart from the case $\lambda = 0$, where no solution exists).