Problem 1.

Consider the unconstrained optimisation problem

$$\min_{x,y} f(x, y),$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x, y) = \frac{1}{4}x^4 + 2x^2y^2 + 4x^2y + 2x^2 + y^2 + 2y.$$

a) Find all local and global minimisers of f.
(10 points)

b) Determine whether the function f is convex.
(5 points)

c) Assume that you want to perform one step of the gradient descent method starting from the point $(x, y) = (0, 0)$. For which step lengths $\alpha > 0$ are the strong Wolfe conditions with parameters $c_1 = 0.1$ and $c_2 = 0.9$ satisfied?
(10 points)

d) Assume you want to solve this optimisation problem with Newton's method, using backtracking Armijo line search. Can you guarantee that this method converges for all initialisations? In case the algorithm converges, what convergence rate do you expect?
(10 points)

Problem 2.

Consider the optimisation problem

$$x + y \longrightarrow \min$$

subject to the constraint $(x, y) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is given by the constraints

$$x^2 + y^2 \geq 25, \qquad y \geq 0, \qquad x \leq 5, \quad \text{and} \quad 3y \leq 4x.$$

a) Sketch the set Ω and determine for each point in Ω whether the LICQ holds.
(5 points)

b) Determine the tangent cone and the cone of linearised feasible directions to the set Ω in the points $(x, y) = (3, 4)$ and $(x, y) = (5, 0)$.
(10 points)

c) Find all KKT points and all local and global minimisers for this optimisation problem.
(15 points)

Problem 3.

Let $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$ be given, and consider the optimisation problem

$$\|x\|_1 \longrightarrow \min \qquad \text{subject to } Ax \geq b. \tag{1}$$

Formulate the Lagrangian dual of (1) as a constrained optimisation problem.
(10 points)

Problem 4.

Let $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$ be given, and assume that $f\colon \mathbb{R}^d \to \mathbb{R}$ is strictly convex, coercive, and continuously differentiable. Assume moreover that the equation $Ax = b$ has some solution $\hat{x} \in \mathbb{R}^d$. We consider the constrained optimisation problem

$$\min_x f(x) \qquad \text{subject to } Ax = b. \tag{2}$$

Moreover, for $\lambda \in \mathbb{R}^m$ and $\mu > 0$, we denote by

$$\mathcal{L}_A(x, \lambda; \mu) = f(x) - \langle \lambda, Ax - b\rangle + \frac{\mu}{2}\|Ax - b\|_2^2$$

the augmented Lagrangian of this problem.

a) Show that the constrained optimisation problem (2) admits a unique solution $x^* \in \mathbb{R}^d$.
(5 points)

b) Show that the optimisation problem

$$\min_{x\in\mathbb{R}^d} \mathcal{L}_A(x, \lambda; \mu) \tag{3}$$

has for every $\lambda \in \mathbb{R}^m$ and every $\mu > 0$ a unique solution $x_{\lambda,\mu}$.
(5 points)

c) Show that $x_{\lambda,\mu} = x^*$, if and only if $Ax_{\lambda,\mu} = b$.
(5 points)

Problem 5.

Define the functions $f_1, f_2\colon \mathbb{R}_{>0} \to \mathbb{R}$,

$$f_1(x) = (x - 1)^2, \qquad f_2(x) = -\ln(x).$$

Find all Pareto-optimal solutions of the multi-criteria optimisation problem

$$\min_{x>0}\big(f_1(x), f_2(x)\big).$$

(10 points)