Exercise #9

March 14, 2023

Problem 1.

In this exercise, we study the Gauß–Newton method for solving the least-squares problem corresponding to the (overdetermined and inconsistent) system of equations

$$\begin{aligned} x + y &= 1, \\ x - y &= 0, \\ xy &= 2. \end{aligned}$$

To that end, we define

$$\begin{aligned} r_1(x, y) &:= x + y - 1, \\ r_2(x, y) &:= x - y, \\ r_3(x, y) &:= xy - 2, \end{aligned}$$

and

$$f(x, y) := \frac{1}{2} \sum_{j=1}^{3} r_j(x, y)^2.$$

We denote moreover by $J = J(x, y)$ the Jacobian of r.

a) Show that the function f is non-convex, but that it has a unique minimiser (x^*, y^*).

b) Show that the matrix $J^T J$ required in the Gauß–Newton method is positive definite for all x, y.

c) Perform one step of the Gauß–Newton method (without line search) for the solution of this least-squares problem. Use the initial value $(x_0, y_0) = (0, 0)$.

Problem 2.

Let

$$f(x) = x_1^4 + 2x_2^4 + x_1 x_2 + x_1 - x_2 + 2.$$

Starting at the point $x_0 = (0, 0)$ compute explicitly one step for the trust region method with the model function $m(p) = f(x_0) + g^T p + \frac{1}{2} p^T B p$, where $g = \nabla f(x_0)$, $B = \nabla^2 f(x_0)$, and the trust region radius $\Delta = 1$.

Problem 3. (Problem 4.1 in N&W)

Let

$$f(x) = 10(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

At $x = (0, -1)$ draw the contour lines of the quadratic model

$$\min_p m(p) = f(x) + \langle \nabla f(x), p \rangle + \frac{1}{2}\langle p, Bp \rangle \qquad \text{subject to } \|p\| \leq \Delta, \tag{1}$$

assuming that B is the Hessian of f. Draw the family of solutions of (1) as the trust region radius varies from $\Delta = 0$ to $\Delta = 2$. Repeat this at $x = (0, 0.5)$.

Problem 4. (Problem 4.5 in N&W)

Let $\phi(\tau) = m_k(\tau p_k^s)$, where $p_k^s = -\frac{\Delta_k}{\|g_k\|} g_k$ and $m_k(p) = f_k + g_k^T p + \frac{1}{2} p^T B p$ with $B \in \mathbb{R}^{d \times d}$ symmetric. Show that the minimizer of $\phi(\tau)$ subject to $\|\tau p_k^s\| \leq \Delta_k$ and $\tau \geq 0$ is given as

$$\begin{cases} 1, & \text{if } g_k^T B_k g_k \leq 0, \\ \min\left(\frac{\|g_k\|^3}{\Delta_k g_k^T B_k g_k}, 1\right), & \text{otherwise.} \end{cases} \tag{2}$$