Exercise #6

February 21, 2023

Problem 1.

Sketch the region $\Omega = \{(x, y) \in \mathbb{R}^2 : y \geq x \text{ and } y^4 \leq x^3\}$ and compute the tangent cone and the set of linearized feasible directions for each point in Ω. For which point in Ω is the LICQ satisfied?

Problem 2.

Assume that one wants to solve the optimisation problem

$$\max_x f(x) \qquad \text{such that} \qquad \begin{cases} c_i(x) = 0 & \text{for all } i \in \mathcal{E}, \\ c_i(x) \geq 0 & \text{for all } i \in \mathcal{I}. \end{cases}$$

How can we modify the KKT conditions such that one obtains (first order) necessary conditions for this maximisation problem?

Problem 3.

Consider the constrained optimization problem

$$\min_{(x,y)} (x^2 + y^2) \qquad \text{such that} \qquad \begin{cases} x + y \geq 1, \\ \quad y \leq 2, \\ \quad y^2 \geq x. \end{cases}$$

a) Formulate the KKT-conditions for this optimization problem.

b) Find all KKT points for this optimization problem.

c) Find all local and global minima for this optimization problem.

(Part b) can be very tedious. One strategy is to consider all possible active sets and determine for each active set whether KKT-points exist. It can also be extremely helpful to sketch the feasible set and the function.)

Problem 4.

Consider the constrained optimization problem

$$\min_{(x,y)} (xy) \qquad \text{such that} \qquad \begin{cases} y \geq x, \\ y^4 \leq x^3. \end{cases}$$

(Note that the constraint set is the same as in Problem 1.)

a) Find all KKT points and local minima for this optimization problem.

b) Compute the critical cone at $(0,0)$ as defined in the lecture and Nocedal & Wright, and show that there exist directions p contained in the critical cone for which $p^T \nabla^2 \mathcal{L}((0,0), \lambda^*) p < 0$.

c) Show that $p^T \nabla^2 \mathcal{L}((0,0), \lambda^*) p \geq 0$ for all vectors p contained in the tangent cone to the feasible set at $(0,0)$.