Exercise #5

February 14, 2023

Problem 1.

Consider the sets $\Omega_1 = \{x \in \mathbb{R}^d : \|x\|_\infty \leq 1\}$ and $\Omega_2 = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$.

a) Show that Ω_1 and Ω_2 are non-empty, closed and convex sets.

b) In dimension $d = 2$, determine the normal and tangent cones to the sets Ω_1 and Ω_2 at the point $x = (1, 0)$. In addition, determine the normal and tangent cones to Ω_1 at the point $(1, 1)$.

c) Show that the projection π_{Ω_2} onto Ω_2 is explicitly given as

$$\pi_{\Omega_2}(z) = \begin{cases} \dfrac{z}{\|z\|_2} & \text{if } \|z\|_2 > 1, \\ z & \text{else.} \end{cases}$$

d) Consider now the case $d = 2$ and let $f(x) = x_1^2 + (x_2 + 2)^2$. Find the global solution of the problem $\min_{x \in \Omega_2} f(x)$. Also, perform one step of the gradient projection method with the step length $\alpha = \frac{1}{2}$ and initial point $x^0 = (1, 1)$.

Problem 2.

Assume that $\Omega \subset \mathbb{R}^d$ is a non-empty, closed and convex set. Show that the projection mapping $\pi_\Omega : \mathbb{R}^d \mapsto \Omega$ is a non-expansive map in the sense that

$$\|\pi_\Omega(x) - \pi_\Omega(y)\|_2 \leq \|x - y\|_2 \ \forall\, x, y \in \mathbb{R}^d.$$

Problem 3.

Let $A \in \mathbb{R}^{m \times d}$ with $m \geq d$ have full rank, let $b \in \mathbb{R}^m$, and let $\Omega \subset \mathbb{R}^d$ be non-empty, convex, and closed. Consider the restricted least squares problem

$$\min_{x \in \Omega} f(x) \qquad \text{with } f(x) = \frac{1}{2}\|Ax - b\|_2^2 \tag{1}$$

and the gradient projection algorithm

$$x^{(k+1)} \leftarrow \pi_\Omega\big(x^{(k)} - \alpha \nabla f(x^{(k)})\big).$$

Show that this algorithm converges to the unique solution of (1) provided that $0 < \alpha < 2/\sigma_{\max}^2$, where $\sigma_{\max}$ denotes the largest singular value of A.

Hint: Show that the gradient descent step $x \mapsto x - \alpha \nabla f(x)$ is a contraction on $\mathbb{R}^d$, and then use the result of the previous exercise and Banach's fixed point theorem.

Problem 4. (Exercise 12.4, N&W Book)

If $f : \mathbb{R}^d \mapsto \mathbb{R}$ is convex and the feasible region Ω is convex, show that local solutions of the problem $\min_{x\in\Omega} f(x)$ are also global solutions. Show that the set of global solutions is convex.

Problem 5.

Consider the set

$$\Omega := \left\{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } x^2(x+1) - y \geq 0\right\}.$$

Determine the tangent cone and the cone of linearized feasible directions to Ω at the points $(x, y) = (-1, 0)$, $\left(-\frac{2}{3}, 0\right)$, and $(0, 0)$.