Exercise #4

February 7, 2023

Problem 1.

Implement Algorithm 5.2 (N&W Book, Page Number 112) and use it to solve linear systems in which A is the Hilbert matrix, whose elements are $A_{i,j} = \frac{1}{i+j-1}$. Set the right-hand-side to $b = (1, 1, \ldots, 1)^T$ and the initial point to $x_0 = 0$. Try dimensions $n = 5, 8, 12, 20$ and report the number of iterations required to reduce the residual below 10^{-6}.

Problem 2.

Implement Algorithm 5.4 (N&W Book, Fletcher-Reeves Method) for the *Rosenbrock* function $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$, where the step length α_k should satisfy the strong Wolfe conditions (5.43) (N&W Book, Page Number 122) with $0 < c_2 < \frac{1}{2}$.

Hint: You can use a bracketing method as discussed in the lecture on January 23 for finding a suitable step length. In this case, you have to recall, though, that the two conditions

$$f(x_k + \alpha p_k) \le f(x_k) + c_1 \alpha \langle \nabla f(x_k), p_k \rangle,$$
$$\langle \nabla f(x_k + \alpha p_k), p_k \rangle \le -c_2 \langle \nabla f(x_k), p_k \rangle,$$

indicate that a step length α is not too large (that is, if one of those is violated, then the current step length *is* too large), while only the weak curvature condition

$$\langle \nabla f(x_k + \alpha p_k), p_k \rangle \ge c_2 \langle \nabla f(x_k), p_k \rangle$$

prevents too small step lengths α.

Alternatively, you can use the algorithm that is described in N&W, Section 3.5.

Problem 3.

Let $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Use the CG-method (Algorithm 5.2, N&W Book) with initialization $x_0 = 0$ for solving the linear system $Ax = b$.

Problem 4.

Show that when applied to a quadratic function, with exact line searches, both the Polak–Ribière formula given by (5.44) (N&W Book) and the Hestenes–Stiefel formula given by (5.46) (N&W Book) reduce to the Fletcher–Reeves formula (5.41*a*) (N&W Book).

Hint: Prove $r_{k+1}^T p_k = 0$, $r_k^T p_k = -r_k^T r_k$ and $r_{k+1}^T r_k = 0$ by using $x_{k+1} = x_k + \alpha_k p_k$, $p_{k+1} = -r_{k+1} + \beta_{k+1} p_k$ and $\alpha_k = -\frac{r_k^T p_k}{p_k^T A p_k}$.

Problem 5.

Prove that Lemma 5.6 (N&W Book, Page Number 125) holds for any choice of β_k satisfying $|\beta_k| \leq \beta_k^{\mathrm{FR}}$ for $k \geq 2$.

Problem 6.

Assume that $m > n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Consider the following algorithm:
Choose $x_0 \in \mathbb{R}^n$ arbitrary, set $r_0 \leftarrow Ax_0 - b,\ s_0 \leftarrow A^T r_0,\ p_0 \leftarrow -s_0,$ and $k \leftarrow 0$.
While $s_k \neq 0$:

$$\alpha_k \leftarrow \frac{\|s_k\|^2}{\|Ap_k\|^2},$$

$$x_{k+1} \leftarrow x_k + \alpha_k p_k,$$

$$r_{k+1} \leftarrow r_k + \alpha_k A p_k,$$

$$s_{k+1} \leftarrow A^T r_{k+1},$$

$$\beta_{k+1} \leftarrow \frac{\|s_{k+1}\|^2}{\|s_k\|^2},$$

$$p_{k+1} \leftarrow -s_{k+1} + \beta_{k+1} p_k,$$

$$k \leftarrow k + 1.$$

Assume that the matrix A has full rank. Show that the above mentioned algorithm is actually identical with the CG-algorithm (Algorithm 5.2 (N&W Book)) for the solution of $A^T Ax = A^T b$ (in the sense that the iterates x_k of both methods coincide).
Hint: Prove $r_{k-1}^{\mathrm{CG}} = s_{k-1},\ p_{k-1}^{\mathrm{CG}} = p_{k-1},\ \alpha_{k-1}^{\mathrm{CG}} = \alpha_{k-1}$ and $x_k^{\mathrm{CG}} = x_k$, forall k by the mathematical induction method.